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Autor(en): **Misra, B. / Sinha, K.B.**

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A remark on the rate of regeneration in decay processes

by **B. Misra**

Institut Solvay, Université Libre de Bruxelles

and **K. B. Sinha***

Département de Physique Théorique, Université de Genève

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Abstract. It is shown that in decay process the regeneration of the undecayed unstable states from the decay products can not proceed too slowly (in a sense specified below) at *small time* and correspondingly the deviation from exponential decay at *small time* can not be too small either. This is seen to be essentially a consequence of the semiboundedness of the Hamiltonian. As by product, we also obtain some sufficient conditions for the existence of certain operator limits that arise in the analysis of continuous observations in quantum mechanics and in the theory of generalized product formulas for semi-groups.

1. Introduction

In this note we consider two related problems: one pertaining to regeneration in the decay process of unstable particles [1–4] and the other arising in the analysis of continuous observation in quantum mechanics [5].

Let $\mathscr{H}$ denote the Hilbert space of state vectors of the decaying system together with its decay products and E denote the projection into the subspace spanned by the undecayed (unstable) states of the system. $E^{\perp} = I - E$ then denotes the projection onto the subspace of 'decayed' states. Time-evolution of the total system is described by a one-parameter unitary group which we denote by $U_t = e^{-iHt}$. As discussed in [3] and [4] absence of regeneration of undecayed unstable states from the decay products is mathematically expressed by the semi-group property of the reduced evolution $Z(t) = EU_tE$:

$$Z(t + s) - Z(t)Z(s) = EU_tE^{\perp}U_sE = 0 \tag{1}$$

for $t, s \geqslant 0$.

Consequences of this assumption (i.e., no regeneration at all time or more generally no regeneration at all time after a finite initial period) are studied [1–4] and it is found that it is in conflict with the physical requirement that the spectrum of the Hamiltonian H, the self adjoint generator of U_t, be bounded below. Here we study the consequence of having non vanishing but a slow rate of regeneration. It is shown that too slow a rate of regeneration at small time (in the sense to be made precise below) is again in conflict with the semiboundedness of the Hamiltonian.

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As a byproduct of this consideration we also obtain some sufficient conditions for the existence of the

$$s - \lim_{n \to \infty} (EU_{t/n}E)^n \equiv T(t); \quad t \geqslant 0 \tag{2}$$

As discussed in [5] study of this limit arises in the analysis of continuous observation in quantum mechanics. Unfortunately, the sufficient conditions obtained here are inapplicable in physically interesting examples but they are given here with the hope that they might suggest suitable generalizations.

2. Rate of regeneration in decay process

The discussion of both the above-mentioned questions is based on the following principal theorem.

Theorem 1.

Let $t \geqslant 0$, $F(t)$, be a strongly continuous contractive family of bounded operators in $\mathscr{H}$ and let for all ψ in some dense set D,

$$\| [F(t+s) - F(t)F(s)]\psi \| \leqslant C_\psi t^\alpha s^\alpha; \tag{3}$$

for $t, s \geqslant 0$, with $\alpha > 1$ and C_ψ a constant independent of t, s but depending on ψ.

Then $F(t)$'s form a strongly continuous semigroup, for $t \geqslant 0$:

$$F(t+s) = F(t)F(s); \quad t, s \geqslant 0 \tag{4}$$

Remark. Since $\| F(t) \| \leqslant 1$ it is evident that the condition (3) is actually a restriction on $F(t)$ for small t and s only, it being automatically satisfied for large t and s.

Proof. Let $\{t_i\}_{i=1}^n$ be a set of positive numbers. Then

$$\begin{aligned}
&\left[F\left(\sum_{i=1}^n t_i\right) - \prod_{i=1}^n F(t_i)\right]\psi \\
&= \left[F\left(t_1 + \sum_{i=2}^n t_i\right) - F(t_1)\prod_{i=2}^n F(t_i)\right]\psi \\
&= \left[F\left(t_1 + \sum_{i=2}^n t_i\right) - F(t_1)F\left(\sum_{i=2}^n t_i\right) + F(t_1)\left\{F\left(\sum_{i=2}^n t_i\right) - \prod_{i=2}^n F(t_i)\right\}\right]\psi
\end{aligned}$$

and therefore using (3) and triangle inequality we obtain,

$$\begin{aligned}
&\left\| \left[F\left(\sum_{i=1}^n t_i\right) - \prod_{i=1}^n F(t_i)\right]\psi \right\| \\
&\leqslant C_\psi t_1^\alpha \left(\sum_{i=2}^n t_i\right)^\alpha + \left\| \left[F\left(\sum_{i=2}^n t_i\right) - \prod_{i=2}^n F(t_i)\right]\psi \right\|
\end{aligned}$$

Iterating this process and in the end substituting $t_i = t/n$ for all i, we are led to

$$\begin{aligned}
&\|[F(t) - F(t/n)^n]\psi\| \\
&\leqslant C_\psi \left(\frac{t}{n}\right)^{2\alpha} \sum_{i=1}^{n-1} i^\alpha \leqslant C_\psi \left(\frac{t}{n}\right)^{2\alpha} \int_1^{n-1} x^\alpha \, dx \\
&= C_\psi \left(\frac{t}{n}\right)^{2\alpha} \frac{(n-1)^{\alpha+1} - 1}{a+1} \leqslant C_\psi \left(\frac{t}{\alpha+1}\right)^{2\alpha} n^{1-\alpha} \ldots
\end{aligned} \tag{5}$$

The inequality (5) thus shows that the sequence of operators $[F(t/n)]^n$ converges (strongly) to $F(t)$ on D as $n \to \infty$. Since the involved family of operators is uniformly bounded ($\|[F(t/n)]^n\| \leqslant 1$) and D is dense in $\mathscr{H}$ it then follows that for $t \geqslant 0$

$$s - \lim_{n \to \infty} F(t/n)^n = F(t) \tag{6}$$

on the whole of $\mathscr{H}$.

Following Chernoff [6] one can now prove the semigroup property of $F(t)$ from the relation (6). In fact, let t and s be two rationally related positive numbers so that there exist positive integers p and q such that $(s + t)/r(p + q) = s/rp = t/rq$ for all integers r. Then

$$[F((s + t)/r(p + q))]^{r(p+q)} = [F(s/rp)]^{rp}[F(t/rq)]^{rq}$$

Going to the limit $r \to \infty$ and using (6) we obtain

$$F(s + t) = F(s)F(t)$$

for all rationally related positive numbers s and t. The same relation then persists for all s, $t \geqslant 0$ in view of the assumed continuity of $F(t)$.

Corollary 1.

If for a dense set of vectors ψ in $\mathscr{H}$

$$\|EU_sE^\perp U_tE\psi\| \leqslant C_\psi t^\alpha s^\alpha; \quad t, s \geqslant 0 \tag{7}$$

with $\alpha > 1$ then

$$EU_sE^\perp U_t = 0 \qquad \text{for all} \quad t, s \geqslant 0;$$

i.e., EU_tE is a semigroup for $t \geqslant 0$.

To interpret this result physically in the context of particle decay let us note that the quantity $\|EU_sE^\perp U_tE\psi\|^2$ represents the probability of regeneration of the undecayed unstable states from the 'decayed' component $E^\perp U_tE\psi$ of the state $U_tE\psi$. Thus the function

$$R(t, s) \equiv \|EU_sE^\perp U_tE\psi\|$$

provides an estimate of (the rate of) regeneration of the undecayed unstable states from the decay products. The preceding corollary, then, shows that too *slow* a rate of regeneration at *small* time characterized by the inequality (7) with $\alpha > 1$ implies that there is in fact no regeneration at all! Absence of regeneration (together with the physical requirement that the Hamiltonian be semibounded) implies, however, that there is no decay (U_t commutes with E) [1–4, see also the corollary in section 3 of (5)]. We may therefore conclude that in particle decay process regeneration must proceed at a sufficiently *rapid* rate at *small* time as implied by the fact that the function $R(t, s)$ introduced above cannot satisfy the bound

$$R(t, s) \leqslant C_\psi t^\alpha s^\alpha \quad \text{for} \quad t, s \geqslant 0 \quad \text{with} \quad \alpha > 1.$$

Since the process of regeneration is believed to be the *physical* mechanism responsible for the deviation from the exponential decay law [cf. 3] this result indicates also that the deviation from exponential decay at *small* time cannot be too small. The present discussion does not provide, however, a quantitative estimate of this deviation.

In this connection it may be recalled that the necessity for deviation from an exponential decay law *at large time* can be ultimately traced, via the Paley-Wiener theorem, to the semiboundedness of the Hamiltonian [cf. 7]. The preceding discussion shows that the deviation at *small time* also can be traced to the same source via theorem 1 and the results of [1–4]. We find that the semiboundedness of the Hamiltonian implies sufficiently appreciable regeneration and correspondingly non negligible deviation from exponential decay at *small time* too. We now turn to the second question.

3. The existence of $s - \lim_{n\to\infty} (EU_{t/n}E)^n$

As mentioned before, the consideration of the above limit arises in the analysis of continuous observations in quantum theory [5]. Apart from this application, considerations of such limits seem to possess intrinsic mathematical interest in the context of extending the Trotter product formula for semigroups [6, 8].

It may be recalled that Trotter's paper [8] studies the existence and properties of

$$s - \lim_{n\to\infty} [F(t/n)]^n \tag{8}$$

where $F(t)$ is of the form

$$F(t) = S(t)T(t)$$

with $S(t)$ and $T(t)$ two contraction valued semigroups for $t \geqslant 0$. With a view to generalizing this analysis Chernoff [6] has investigated the limit (8) when $F(t)$ is any strongly continuous contraction valued function. The limit (2) under discussion here is a special important instance of this generalization.

The known sufficient conditions [6] for the existence of limit (8) are expressed in terms of the properties of $F'(0)$, the derivative of $F(t)$ at $t = 0$, which are difficult to verify in concrete situations of physical interest. We mention here two other sufficient conditions that are immediately suggested by the preceding discussion.

One is the condition (7) (with $\alpha > 1$) of the corollary stated before. But this condition implies also that EU_tE is itself a semigroup for $t \geqslant 0$ so that it applies only either in the trivial situation that E commutes with U_t or in the physically uninteresting situation that the Hamiltonian H, the generator of U_t, is not semibounded. It is thus desirable to search for conditions that guarantee the existence of the limit (2) without at the same time implying that EU_tE is itself a semigroup. One candidate for such a condition is again the inequality (7) but with $\alpha = 1$. We are, however, not able to decide the truth or falsity of this guess at present but mention the following result in this direction.

Theorem 2.

Let

$$\|EU_{t+s}E - EU_tEU_sE\| \leqslant Cts, \quad t, s \geqslant 0 \cdots \tag{6}$$

Then

$$(EU_{t/n}E)^n \text{ converges in norm.} \tag{7}$$

Proof. As in the proof of the theorem, substituting $\alpha = 1$, we conclude that

$$\|EU_tE - (EU_{t/n}E)^n\| \leqslant Ct^2$$

and therefore,

$$\|(EU_{t/m}E) - (EU_{t/mn}E)^n\| \leqslant C(t/m)^2$$

Now

$$(EU_{t/m}E)^m - (EU_{t/mn}E)^{nm}$$

$$= \sum_{j=0}^{m-1} (EU_{t/m}E)^j\{(EU_{t/m}E) - (EU_{t/mn}E)^n\}(EU_{t/mn}E)^{n(m-j-1)}$$

Using the triangle inequality and the uniform boundedness of EU_tE we obtain

$$\|(EU_{t/m}E)^m - (EU_{t/mn}E)^{nm}\| \leqslant Ct^2/m$$

Interchanging m and n, we are led to

$$\|(EU_{t/n}E)^n - (EU_{t/mn}E)^{nm}\| \leqslant Ct^2/n$$

Therefore,

$$\|(EU_{t/m}E)^m - (EU_{t/n}E)^n\| \leqslant Ct^2\left(\frac{1}{n} + \frac{1}{m}\right) \to 0$$

as $n, m \to \infty$. Hence the conclusion follows.

However, once again, this condition is not sufficiently general to settle the question of existence of the limit in physically interesting situations. In particular, the question of existence of the limit (2) in the case when U_t is the unitary evolution group of a free particle generated by the free Hamiltonian $-\Delta$ in $L^2(R^3)$ and E is the projection on to the subspace of functions with support in a given finite region of R^3 seems to be still open [cf. 9].

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