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Boltzmann collision operator without cut-off [1])

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Abstract. We study the spectrum of the linear Boltzmann collision operator for repulsive inverse-power intermolecular potentials. The spectrum turns out to be purely discrete. We use the method of strong resolvent convergence and study various cut-off approximations. A previous work of Pao [1] is critically reviewed.

1. Introduction

In a recent work of Pao [1] the linear Boltzmann collision operator for gases with repulsive inverse-power intermolecular potential was considered. In contrast to almost all of the previous work on this subject *no* cut-off to exclude the grazing collisions was introduced. On the one hand, it is well known that, due to the grazing collisions, infinite range potentials show up a grave singularity in their classical differential cross section. On the other hand, the collision operator has built in a mechanism which cancels this singularity to some degree. Though a radial cut-off of the intermolecular potential or even the use of a quantum mechanical cross section may be physically reasonable, there has been a constant interest in the spectral properties of the collision operator without cut-off. Since the inverse fourth-power potential (Maxwell gas) has a purely discrete spectrum, the question is natural whether a similar result holds for other potentials. According to Pao this is true for potentials with power $s > 2$. Pao uses the theory of pseudo-differential operators, a method which seems to be adequate to the problem but which also requires some tedious estimates of the so-called symbols of these pseudo-differential operators. Moreover, in our opinion, Pao's discussion of the selfadjointness of the collision operator is erroneous[2]). It seems, therefore, to be justified to attack the problem by another method. We present some relatively simple arguments which prove the discreteness of the spectrum for inverse-power potentials. In particular, we construct the collision operator for the infinite range potential as the limit of operators with a cut-off, for example, an angular (Grad) cut-off. The approximation will be in the

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[1]) Work supported by the Swiss National Science Foundation.
[2]) See Note on p. 903.

sense of strong resolvent convergence. Among the various kinds of cut-offs, the radial integral cut-off [3], which is closely related to a radial potential cut-off ([4], [3]), has some unpleasant properties. However, we can also describe the spectrum in this case. Although we confine ourselves to the study of strict inverse-power potentials it becomes evident that our method can also be used in other cases.

2. Preliminaries

The linearized Boltzmann equation for a spatially homogeneous gas reads in nondimensional form

$$\frac{\partial f}{\partial t} = If \tag{2.1}$$

where

$$(If)(\mathbf{v}) = \int (f(\mathbf{v}_1') + f(\mathbf{v}') - f(\mathbf{v}_1) - f(\mathbf{v}))\omega(v_1)B(\theta, V)\, d^3v_1\, d\theta\, d\varphi \tag{2.2}$$

$$\omega(v_1) = (2\pi)^{-3/2} \exp\left(-\frac{v_1^2}{2}\right) \tag{2.3}$$

Here $\mathbf{v}$, $\mathbf{v}_1$ are the velocities before and $\mathbf{v}'$, $\mathbf{v}_1'$ the velocities after an encounter:

$$\mathbf{v}' = \mathbf{v} + \mathbf{n}(\mathbf{n}\mathbf{V}) \tag{2.4}$$

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{n}(\mathbf{n}\mathbf{V}) \tag{2.5}$$

$$\mathbf{n}\mathbf{V} = V\cos\theta \tag{2.6}$$

$$\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{2.7}$$

$$\mathbf{V} = \mathbf{v}_1 - \mathbf{v} \tag{2.8}$$

$$B(\theta, V) = Vb(\theta, V)\left|\frac{\partial b}{\partial \theta}(\theta, V)\right|. \tag{2.9}$$

φ is a polar angle in the impact parameter plane, $0 \leq \varphi \leq 2\pi$. $B(\theta, V)$ is related to the differential cross section and b denotes the impact parameter [2]. For analytical purposes, it is an advantage to assume a pure power law interaction

$$U(r) = \frac{\alpha}{r^s} \quad \alpha > 0, s > 0. \tag{2.10}$$

Then $\theta \in [0, \pi/2]$, $b \in [0, \infty)$. Further restrictions on s will follow. We have

$$B(\theta, V) = V^\gamma \beta(\theta), \quad \beta(\theta) \geq 0, \quad \gamma = \frac{s-4}{4} \tag{2.11}$$

$$\beta(\theta) \sim \theta \text{ as } \theta \to 0 \quad \text{and} \quad \beta(\theta) \sim \left(\frac{\pi}{2} - \theta\right)^{-1-2/s} \text{ as } \theta \to \frac{\pi}{2}. \tag{2.12}$$

If we would have, for a general interaction $U(r)$,

$$\int_0^{\pi/2} B(\theta, V)\, d\theta < \infty$$

we could write

$$I = -\nu + K \tag{2.13}$$

splitting away the multiplicative operator $\nu = \nu(v)$, the so-called collision frequency.

In order to have (2.13), one must introduce a cut-off. Some possibilities are

(a) *angular* cut-off: Set $B(\theta, V) = 0$ if $\theta > \theta_0$ for some θ_0, $0 < \theta_0 < \pi/2$, uniformly in V.
(b) *radial potential* cut-off: Set $V(r) = 0$ for $r > \sigma$.
(c) *radial integral* cut-off: replace the θ-integration in (2.2) by an integration over b and cut off this integral at some value b_0.

(a) was introduced by Grad[2]. In this case, K is compact whenever $s > 1$ [2], [3]. (b) was discussed by Cercignani [4] and (c) appears in Drange [3]. For (b) and (c), K is *not* compact [3].

We consider the operator I in the Hilbert space $\mathscr{H}$ with scalar product

$$(f, g) = \int \omega f \bar{g} d^3v, \qquad \mathscr{H} = L_2(\mathbb{R}^3, \omega\, d^3v). \tag{2.14}$$

It is straightforward to define If for smooth functions f because it is possible to carry out first the θ-integration in (2.2). By (2.4)–(2.8) we get for θ near $\pi/2$

$$(f(\mathbf{v}_1') + f(\mathbf{v}') - f(\mathbf{v}_1) - f(\mathbf{v})) \sim \left(\frac{\pi}{2} - \theta\right). \tag{2.15}$$

We see from (2.12) that $s > 2$ is necessary for the θ-integration to exist. In fact, the matrix elements of the collision operator with respect to the so-called Burnett functions (eigenfunctions of the Maxwell gas) are finite only if $s > 2$ [6].

Using the symmetry properties of the differential cross section one shows that

$$(f, If) = -\int |f(\mathbf{v}_1') + f(\mathbf{v}') - f(\mathbf{v}_1) - f(\mathbf{v})|^2 \times$$
$$\times\, \omega(v_1)\omega(v) B(\theta, V)\, d\theta\, d\varphi\, d^3v_1\, d^3v \leq 0. \tag{2.16}$$

I has a five-fold zero eigenvalue with eigenfunctions $1, v_i, v^2$.

3. The spectrum of I

We study first the case of a *hard* intermolecular potential, i.e. an inverse-power law with $s \geq 4$.

We introduce an angular cut-off, setting $\beta(\theta) = 0$ for $\theta > \pi/2 - \delta$ with some small positive δ. The corresponding collision operator is denoted by I_δ. We are interested in the limit $\delta \downarrow 0$. I_δ consists of two parts [2]

$$I_\delta = -\nu_\delta + K_\delta \tag{3.1}$$

$$\text{with } \nu_\delta(v) = (2\pi)\beta_\delta \int \omega(v_1)|\mathbf{v}_1 - \mathbf{v}|^\gamma\, d^3v_1 \quad \gamma = \frac{s-4}{s} \tag{3.2}$$

$$\beta_\delta = \int_0^{\pi/2-\delta} \beta(\theta)\, d\theta \tag{3.3}$$

$\gamma > 0$ if $s > 4$ and $\nu_\delta(v)$ grows like v^γ as $v \to \infty$. In fact, $\nu_\delta(v)$ is a monotonically increasing function starting from some positive value $\nu_\delta(0)$. For $s = 4$, ν_δ is a constant. K_δ is a compact integral operator [2]. I_δ is selfadjoint and negative on $D(I_\delta) = D(\gamma_\delta) = D(v^\gamma)$ (independent of δ).

Our aim is to apply Theorem (3.13) of Kato ([5], p. 459). This theorem is formulated in terms of sesquilinear forms and uses the close connection between forms and operators. To I_δ corresponds a closed, symmetric, negative quadratic form, denoted by $I_\delta[\cdot]$ with

$$i_\delta[f] = -\| |I_\delta|^{1/2} f \|^2 \quad f \in D(|I_\delta|^{1/2}) \tag{3.4}$$

Consider now $I \upharpoonright C_0^\infty(\mathbb{R}^3)$ (I restricted to the C^∞-functions with compact support). $I \upharpoonright C_0^\infty$ defines a closable form since it is generated by a symmetric negative operator. We denote the closure of this form by i_F. The index 'F' stands for 'Friedrichs' and means that the operator I_F associated with i_F is the Friedrichs extension of $I \upharpoonright C_0^\infty$. I_F is selfadjoint and negative.

For two selfadjoint, negative operators A and B the order relation $A \leq B$ is defined as equivalent to $a \leq b$ for the associated closed forms a and b, which means that $D(a) = D(|A|^{1/2}) \subset D(b)$ and $a[f] \leq b[f]$ for all $f \in D(a)$.

Proposition 3.1. $I_{\delta_1} \geq I_{\delta_2} \geq I_F$ if $0 < \delta_2 \leq \delta_1$.

Proof: In view of (2.12), (2.16) and (3.3) the inequalities certainly hold for $f \in C_0^\infty$ since $\beta_{\delta_2} \geq \beta_{\delta_1}$. Then Proposition 3.1 follows by taking closures of quadratic forms using the fact that I_δ is essentially selfadjoint on C_0^∞ (because ν_δ is essentially selfadjoint).

Now the theorem from Kato's book quoted above gives

Theorem 3.2. *I_δ converges in the strong resolvent sense to a selfadjoint, negative operator I_0*

$$(I_\delta - \lambda)^{-1} f \to (I_0 - \lambda)^{-1} f \quad \textit{for } \operatorname{Re}\lambda > 0, f \in \mathscr{H}$$

and

$$I_0 \leq I_\delta.$$

I_0 is the collision operator with the angular cut-off removed. There exists another characterization of I_0. I_0 is also the strong graph limit of I_δ ([7], p. 293). This means that $f \in D(I_0)$ if and only if we can find $f_\delta \in D(I_\delta)$ so that $f_\delta \to f$ and $I_\delta f_\delta$ converges strongly to some element g. Then $g = I_0 f$.

Remarks

(1) The Friedrichs extension I_F of $I \upharpoonright C_0^\infty$ acts only as an auxiliary operator and is used in the proof of Theorem 3.2. Instead of C_0^∞, we could, for example, use $\mathscr{P}$, the class of polynomials in v_i. $\mathscr{P}$ would contain the Burnett functions.

(2) One can show that $I_\delta f$ converges strongly when $f \in C_0^\infty$ or $f \in \mathscr{P}$. Hence $I \upharpoonright C_0^\infty \subset I_0$ and $I \upharpoonright \mathscr{P} \subset I_0$. For the proof it is important to notice that the singularity in the collision operator at $\theta = \pi/2$ is cancelled by the smoothness of f. This fact allows us to estimate the differences $f(\mathbf{v}_1') - f(\mathbf{v}_1)$ and $f(\mathbf{v}') - f(\mathbf{v})$ by means of the gradient of f. It is then easy to show that $I_\delta f$ converges exactly to that function

which is defined by (2.2), integrating in the ordinary sense. For f one has to use the fact that the gradient is also polynomially bounded.

(3) $\exp(tI_\delta) \to \exp(tI_0)$ strongly for each $t \geq 0$ ([7], p. 286).

It is our intention to show that $(I_0 - \lambda)^{-1}$ is compact. This looks somewhat surprising since $(I_\delta - \lambda)^{-1}$ is not compact and we have only proved strong convergence of the resolvents. For $a, b \in \mathbb{R}$ we write $P_\delta[a, b]$ for the spectral projection of the operator I_δ associated with the interval $[a, b]$.

Proposition 3.3. *To a given $N > 0$ there exists a $\delta_N > 0$, so that*

$$\sigma_{\text{ess}}(I_\delta) \cap [-N, 0] = \emptyset$$

and

$$\dim P_\delta[-N, 0] < \infty \quad \text{for } 0 \leq \delta < \delta_N.$$

The bound on $\dim P_\delta[-N, 0]$ *is independent of* δ.

Proof: By Weyl's theorem ([5], p. 244) and the compactness of K_δ, the spectrum of I_δ in the gap $(-v_\delta(0), 0]$ consists of discrete eigenvalues with finite multiplicities, and the essential spectrum is the interval $(-\infty, -v_\delta(0)]$. By (3.2) $v_\delta(0) \sim \beta_\delta \to \infty$ as $\delta \downarrow 0$ and the statement about σ_{ess} follows, at least for $\delta_N > \delta > 0$. However from Theorem 3.2 we know that $I_0 \leq I_\delta$ and this implies $\dim P_0[-N, 0] \leq \dim P_\delta[-N, 0] \leq \dim P_{\delta_N}[-N, 0]$. For suppose $\dim P_\delta > \dim P_0$. We can find $f \in \operatorname{Ran} P_0 \cap \operatorname{Ran}(1 - P_\delta)$. Then $f \in D(I_0) \subset D(|I_0|^{1/2}) \subset D(|I_\delta|^{1/2})$ and $-N \leq (f, I_0 f) \leq (f, I_\delta f) < -N$, and this is a contradiction.

It follows from Proposition 3.3 that I_0 has a discrete spectrum consisting of eigenvalues with finite multiplicities. There is actually an infinite sequence of eigenvalues tending to $-\infty$. Otherwise I_0 would be bounded and this is not true. Consider, for instance, the matrix elements with respect to Burnett functions [6].

We can summarize these properties in

Theorem 3.4. $(I_0 - \lambda)^{-1}$ *is compact for* $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ *and* $\dim P_0[-N, 0] \to \infty$ *as* $N \to \infty$.

Compactness of the resolvent in the region indicated above follows from compactness for $\lambda > 0$ and by analytic continuation.

Remarks

(1) The eigenvalues of I_δ converge from above to the eigenvalues of I_0.

(2) $P_\delta[-N, 0] \to P_0[-N, 0]$ strongly but also in norm ([5], p. 438).

Let us now consider the case of a *soft* power law potential obeying $2 < s < 4$.

We want to show how our method works in this case. We shall not prove all technical details.

For an angular cut-off it is still true that K_δ is a compact operator. But since $\gamma < 0$

$$v_\delta(v) \to 0, \quad v \to \infty \tag{3.5}$$

and

$$\sigma_{\text{ess}}(I_\delta) = [-\nu_\delta(0), 0] \tag{3.6}$$

so that the essential spectrum of I_δ is not shifted to $-\infty$ as $\delta \downarrow 0$ but fills up larger and larger intervals on the negative axis. Consequently, the proofs of Proposition 3.3 and Theorem 3.4 break down. However, the arguments behind Theorem 3.2 remain true and the strong resolvent limit exists. It is possible that the limiting operator has a discrete spectrum because the spectrum of I_δ can contract suddenly ([7], p. 291). We will see that this happens indeed. But for a discussion of the limiting operator the angular cut-off is not the right choice.

Let us consider briefly a radial integral cut-off as it was described in section 2. We set $b_0 = \sigma$ for the cut-off distance. The collision frequency is of the form

$$\nu_\sigma(0) > 0, \qquad \frac{d\nu_\sigma(v)}{dv} \geq 0, \qquad \nu_\sigma(v) \sim \sigma^2 . v, \quad v \to \infty. \tag{3.7}$$

Actually, $\nu_\sigma(v)$ is the collision frequency of hard spheres with diameter σ [2].

We notice that a radial integral cut-off can also be considered as a velocity-dependent angular cut-off with cut-off angle $\theta_0 = \theta_0(V, \sigma)$. It follows from (2.9), (2.11) and (2.12) that

$$\frac{\partial \theta_0}{\partial V} > 0 \tag{3.8}$$

$$\theta_0(V, \sigma) \sim V^{2/s}, \quad V \to 0 \tag{3.9}$$

$$\frac{\pi}{2} - \theta_0(V, \sigma) \sim V^{-2}, \quad V \to \infty. \tag{3.10}$$

We see that, for large V, the small angle scattering is included with increasing accuracy. The noncompactness of K_σ is due to this velocity-dependence, in particular, to the strong decay of the right hand side in (3.10).

A radial potential cut-off would lead to the same difficulties [3]. Instead of the radial integral cut-off we study now a modified velocity-dependent cut-off which excludes the high energy collisions in a more stringent way, but not as drastically as a Grad (velocity-independent) cut-off would do. We define the cut-off angle $\theta(V, \varepsilon)$ through

$$\theta(V, \varepsilon) = \tilde{\beta}^{[-1]}(V^{-\gamma}\varepsilon^{-1}), \quad \varepsilon > 0. \tag{3.11}$$

ε is the cut-off parameter. We also have $\theta_0(V, \sigma) = \tilde{\beta}^{[-1]}(\frac{1}{2}V^{1-\gamma}\sigma^2)$. $\tilde{\beta}^{[-1]}$ is the inverse function of

$$\tilde{\beta}(\theta) = \int_0^\theta \beta(\theta')\, d\theta', \quad \theta \in \left[0, \frac{\pi}{2}\right]. \tag{3.12}$$

(2.12) and (3.12) give

$$\frac{d\tilde{\beta}^{[-1]}}{dz} > 0 \tag{3.13}$$

$$\tilde{\beta}^{[-1]}(z) \sim \sqrt{z}, \quad z \to 0$$

$$\frac{\pi}{2} - \tilde{\beta}^{[-1]}(z) \sim z^{-s/2}, \quad z \to \infty. \tag{3.14}$$

The property which should be compared with (3.10) reads

$$\frac{\pi}{2} - \theta(V, \varepsilon) \sim V^{-2+s/2}, \quad V \to \infty. \tag{3.15}$$

The deviation expressed by (3.15) depends now on s and is bigger than in (3.10).

For the collision frequency we find

$$\nu_\varepsilon(v) = 2\pi \int \omega(v_1) V^\gamma \tilde{\beta}(\theta(V, \varepsilon))\, d^3v_1 = 2\pi\varepsilon^{-1} \int \omega(v_1)\, d^3v_1 = 2\pi\varepsilon^{-1}, \tag{3.16}$$

and this is simply a constant with respect to v. Of course, we have chosen our cut-off such as to get this simplification. Compared with the radial integral cut-off we have sacrificed the growth of the collision frequency for $v \to \infty$, but we have gained the compactness of the integral operator K_ε. The verification of this fact requires estimates of certain integrals using the properties of the function $\tilde{\beta}^{[-1]}$. If we transform from $\mathscr{H}$ to an ordinary $L_2(\mathbb{R}^3)$-space, making the substitution $\hat{f} = \omega^{1/2} f$, we get for that part of the transformed operator $\hat{K}_\varepsilon$ which incorporates all difficulties, an integral kernel of the form [3].

$$\hat{K}_\varepsilon(\mathbf{v}, \mathbf{v}') \sim \frac{1}{u^2} \exp\left[-\tfrac{1}{8}(u^2 + 4\xi_1^2)\right] \int Q(u, w) \exp\left(-\tfrac{1}{2}|\mathbf{w} + \boldsymbol{\xi}_2|^2\right) d\mathbf{w} \tag{3.17}$$

with

$$Q(u, w) = \tfrac{1}{2}(u^2 + w^2)^{\gamma + 1/2} w^{-1} \left[\beta_{\varepsilon, V}(\theta) + \beta_{\varepsilon, V}\left(\frac{\pi}{2} - \theta\right)\right] \tag{3.18}$$

$$\tan\theta = \frac{w}{u}, \qquad V = (u^2 + w^2)^{1/2}, \qquad \mathbf{u} = \mathbf{v} - \mathbf{v}' \tag{3.19}$$

$$\boldsymbol{\xi} = \tfrac{1}{2}(\mathbf{v} + \mathbf{v}') = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \; \boldsymbol{\xi}_1 /\!/ \mathbf{u}, \; \boldsymbol{\xi}_2 \perp \mathbf{u}, \tag{3.20}$$

$$\xi_1^2 = \frac{1}{4} \frac{(v^2 - v'^2)^2}{u^2}$$

$$\begin{aligned} \beta_{\varepsilon, V}(\theta) &= \beta(\theta) & \theta &< \theta(V, \varepsilon) \\ &= 0 & \theta &> \theta(V, \varepsilon) \end{aligned} \tag{3.21}$$

The integration in (3.17) is in a plane perpendicular to $\mathbf{u}$.

If we insert the two summands of (3.18) into (3.17) and decompose the integral into two parts according to $w \geq u$ and $w \leq u$ we get four integrals which have to be estimated. We only give the estimates for one integral, the others can be treated similarly. Consider for $w \geq u$ the expression

$$Q^{(1)}(u, w) = \tfrac{1}{2}(u^2 + w^2)^{\gamma + 1/2} w^{-1} \beta_{\varepsilon, V}(\theta). \tag{3.22}$$

Since $w \geq u$, $\tan\theta > 1$ so that we should have $\tan\theta(V, \varepsilon) > 1$ or $V \geq V_0(\varepsilon) > 0$. It follows from (2.12), (3.11) and (3.14) that

$$\beta(\theta) \leq \beta(\theta(V, \varepsilon)) \leq c\left(\frac{\pi}{2} - \theta(V, \varepsilon)\right)^{-1-2/s} \leq c(u^2 + w^2)^{1/2(2 - s/2 - \gamma)}$$

and hence

$$Q^{(1)}(u, w) \leq c(u^2 + w^2)^{3/2 - s/4} w^{-1} \leq c w^{2 - s/2} \tag{3.23}$$

c denotes a positive constant which is not the same in all inequalities. For $2 < s < 4$ we have $0 < 2 - s/2 < 1$. This implies that the integral in (3.17) is bounded by

$$\text{const}\,(v^{\alpha} + v'^{\alpha}) \quad \alpha = 2 - \frac{s}{2}. \tag{3.24}$$

With this result we can prove the following properties of $\hat{K}_\varepsilon$

$$\sup_{\mathbf{v}\in\mathbb{R}^3} \int |\hat{K}_\varepsilon(\mathbf{v}, \mathbf{v}')|\, d^3v' < \infty \tag{3.25}$$

$$\sup_{\mathbf{v}\geq R} \int |\hat{K}_\varepsilon(\mathbf{v}, \mathbf{v}')|\, d^3v' \to 0, \quad R \to \infty. \tag{3.26}$$

These two statements follow from estimates given in [2] and are due to the fact that $\alpha < 1$, which is the crucial property.

We can now show that $\hat{K}_\varepsilon$ is compact. Firstly, the polar singularity which can be shown to be of the form $1/|\mathbf{v} - \mathbf{v}'|^{\sigma}$ with $\sigma < 3$ can be treated by a limiting argument. We only need to show that

$$\hat{K}_\varepsilon^{(n)}(\mathbf{v}, \mathbf{v}') = K_\varepsilon(\mathbf{v}, \mathbf{v}')\left(1 - \chi\left(\frac{|\mathbf{v} - \mathbf{v}'|}{n}\right)\right)$$

is compact because

$$\hat{K}_\varepsilon(\mathbf{v}, \mathbf{v}') = \operatorname*{norm-lim}_{n\to\infty} \hat{K}_\varepsilon^{(n)}(\mathbf{v}, \mathbf{v}').$$

χ is the characteristic function of $[0, 1]$.

Secondly, the operator $P_R\hat{K}^{(n)}$ with $P_R f = \chi(v/R)f(\mathbf{v})$ is easily seen to be Hilbert–Schmidt. We must therefore only show that

$$\hat{K}_\varepsilon^{(n)} = \operatorname*{norm-lim}_{R\to\infty} P_R\hat{K}_\varepsilon^{(n)}.$$

This can be done as follows (we omit the indices ε and n)

$$|\hat{K}f(\mathbf{v})| \leq \int |\hat{K}(\mathbf{v}, \mathbf{v}')|^{1/2}|\hat{K}(\mathbf{v}, \mathbf{v}')|^{1/2}|f(\mathbf{v}')|\, d^3v'$$

and by the Schwarz inequality

$$|\hat{K}f(\mathbf{v})|^2 \leq \left(\int |\hat{K}(\mathbf{v}, \mathbf{v}')|\, d^3v'\right)\left(\int |\hat{K}(\mathbf{v}, \mathbf{v}')|\, |f(\mathbf{v}')|^2\, d^3v'\right).$$

Using (3.25), (3.26) and the symmetry of $\hat{K}$ we obtain

$$\|(1 - P_R)\hat{K}f\|^2 \leq \left(\sup_{\mathbf{v}\geq R} \int |\hat{K}(\mathbf{v}, \mathbf{v}')|\, d^3v'\right)\left(\sup_{\mathbf{v}'\in\mathbb{R}^3} \int |\hat{K}(\mathbf{v}, \mathbf{v}')|\, d^3v\right)\|f\|^2 \to 0, \quad R \to \infty.$$

We can now investigate the limit $\varepsilon \downarrow 0$ of the operator $I_\varepsilon = -v_\varepsilon + K_\varepsilon$. By (3.11), (3.12) and (3.16) we have

$$\frac{\partial\theta(V, \varepsilon)}{\partial\varepsilon} < 0 \tag{3.27}$$

$$\lim_{\varepsilon \downarrow 0} \theta(V, \varepsilon) = \frac{\pi}{2}, \text{ uniformly in } V \text{ in any interval } V_0 \leq V \leq \infty, \quad V_0 > 0. \tag{3.28}$$

$$v_\varepsilon \sim \varepsilon^{-1} \to \infty, \quad \varepsilon \downarrow 0. \tag{3.29}$$

These results enable us to apply the theorems which we proved for hard potentials. We only have to replace δ by ε.

We can define the strong resolvent limit I_0 of I_ε. Again, I_0 has compact resolvent.

One can ask whether the ordinary (velocity-independent) Grad angular cut-off and our modified velocity-dependent angular cut-off lead to the same limiting operator. This is a uniqueness question and is discussed in the next section.

4. Uniqueness and essential selfadjointness

The method of strong resolvent convergence is useful to construct a selfadjoint limiting operator but gives no answer to the question whether different choices of cut-offs lead to the same limiting operator. Typically, in another field of interest, namely the theory of Dirac operators, the method has been used to construct distinguished selfadjoint extensions of a certain symmetric (in this case not semi-bounded) operator [8].

To prove equivalence of different cut-offs, that is uniqueness of the limiting operator, needs some more work. Indeed, we can show, for example, that Grad's cut-off and a velocity-dependent cut-off give rise to the same limiting operator. We introduce an additional cut-off by restricting V to an interval $[1/M, M]$, $M > 0$.

In the following, the indices δ resp. ε refer to Grad's resp. our velocity-dependent cut-off for a soft potential. The index M refers to the cut-off in V. For $\varepsilon > 0$ we can find $\delta, \delta' > 0$ so that

$$\frac{\pi}{2} - \delta \leq \theta(V, \varepsilon) \leq \frac{\pi}{2} - \delta' \tag{4.1}$$

and to $\delta > 0$ we find $\varepsilon, \varepsilon'$ so that

$$\theta(V, \varepsilon) \leq \frac{\pi}{2} - \delta \leq \theta(V, \varepsilon') \tag{4.2}$$

uniformly for $V \in [1/M, M]$. This follows from (3.28).

Denoting the resolvents (at a point $\lambda > 0$) of $I_{\delta, M}$ resp. $I_{\varepsilon, M}$ by $R_{\delta, M}$ resp. $R_{\varepsilon, M}$ it follows from the above results that

$$\lim_{\delta \downarrow 0} (f, R_{\delta, M} f) = \lim_{\varepsilon \downarrow 0} (f, R_{\varepsilon, M} f) = (f, R_{0, M} f), \quad f \in \mathscr{H}. \tag{4.3}$$

$R_{0, M}$ is the resolvent of a strong resolvent limit $I_{0, M}$. From (4.3) it follows that $R_{\delta, M}$ resp $R_{\varepsilon, M}$ converge strongly to $R_{0, M}$ ([5], p. 452). $I_{0, M}$ is independent of the angular cut-off.

In the limit $M \to \infty$, we get, using $I_{0, M} \leq I_{0, M'}$ if $M \geq M'$, an operator $I_{0, \infty}$. If we first remove the V-cut-off we have $I_{\delta, M} \to I_\delta$ and $I_{\varepsilon, M} \to I_\varepsilon$ in the strong resolvent sense. As δ resp $\varepsilon \downarrow 0$, we get operators I_0' resp I_0. We can now estimate the difference $R_{0, \infty} - R_0$ as follows

$$\begin{aligned} \|(R_{0, \infty} - R_0) f\| \leq \|(R_{0, \infty} - R_{0, M}) f\| + \|(R_{0, M} - R_{\varepsilon, M}) f\| + \\ \|(R_{\varepsilon, M} - R_\varepsilon) f\| + \|(R_\varepsilon - R_0) f\|, \quad f \in \mathscr{H}. \end{aligned} \tag{4.4}$$

To any given $\varepsilon' > 0$ we can find an $\varepsilon > 0$ and $M > 0$, so that each one of the four norms on the right hand side of (4.4) is less than $\varepsilon'/4$. Similarly, we can estimate $R_{0,\infty} - R'_0$. Thus $I_{0,\infty} = I_0 = I'_0$.

Another approach to the uniqueness question would be to prove essential self-adjointness of I on a certain domain of smooth functions, say on C_0^∞ or $\mathscr{P}$. From Pao's work we can deduce that this property should hold on $\mathscr{S}$. $\mathscr{S}$ are the ordinary elements of the Schwartz space transformed to $\mathscr{H}$, that is, multiplied by $\omega^{-1/2}$. This is now the point where we do not quite understand the work of Pao. On p. 579 it is claimed that I is essentially selfadjoint since it is symmetric and negative. However, this is false. There could be a lot of selfadjoint extensions and not all of them bounded from above! So far as we can see through the work of Pao it follows that $\bar{I}$ (closure of $I \upharpoonright \mathscr{S}$) has a compact resolvent and that its eigenfunctions belong to $\mathscr{S}$. But whether $\bar{I}$ is selfadjoint and the eigenfunctions form a complete set seems not to be evident. Pao decomposes $\bar{I}$ with respect to spherical harmonics: $\bar{I} = \bigoplus_{l=0}^{\infty} \bar{I}_l$. He constructs then a compact, 'approximate' inverse B_l of $\bar{I}_l - N$ ($N > 0$, large enough) satisfying

$$B_l(\bar{I}_l - N) \subset 1 + R_l. \tag{4.5}$$

R_l is bounded with $\|R_l\| \leq \text{const}/N$, so that $1 + R_l$ has a bounded inverse if N is large enough. Unfortunately, Pao writes '$=$' instead of '$\subset$'. It follows from (4.5) that (Null $B_l = \{0\}$)

$$(\bar{I}_l - N)^{-1} \subset (1 + R_l)^{-1} B_l \tag{4.6}$$

which means that both sides are equal when acting on elements in Ran $(\bar{I}_l - N)$. For $\bar{I}_l - N$ to be selfadjoint it is necessary and sufficient that Ran $(\bar{I}_l - N) = \mathscr{H}$, or equivalently $\overline{\text{Ran } (I_l - N) \upharpoonright \mathscr{S}_l} = \mathscr{H}_l$.

A sufficient condition for the second statement would be Ran $((1 + R_l)^{-1} B_l \upharpoonright \mathscr{S}_l) \subset \mathscr{S}_l$ so that $\mathscr{S}_l \subset$ Ran $(I_l - N)$. We have not proved this. It is likely, but should be verified, that it follows from the properties of the symbols associated with R_l and B_l in the language of pseudo differential operators used by Pao.[3])

Once the essential selfadjointness on $\mathscr{S}$, or even better on C_0^∞ or $\mathscr{P}$, is established, one can expect that it is no hard problem to show that, by the method of strong resolvent convergence, one actually constructs a selfadjoint extension, which is then unique. For all limiting operators which we have met so far one can show that C_0^∞ or $\mathscr{P} \subset D(I_0)$ (see also a remark in section 3). It is likely that a similar result holds for $\mathscr{S}$.

5. The radial integral cut-off reobserved

We have discarded the radial integral cut-off because it had some unruly properties. Now, however, we can make some statements about spec(I_σ). As before, we consider the case of a soft potential, but it would be possible to treat hard potentials in a similar way. First, $I_\sigma = -\nu_\sigma + K_\sigma$ can be defined as a selfadjoint operator, for example, as it will be done below as a strong resolvent limit of certain cut-off operators.

[3]) See Note on p. 903.

For σ fixed, we introduce a cut-off according to

$$\tilde{\theta}(V, \varepsilon) = \theta_0(V, \sigma) \qquad V < 2\sigma^{-2}\varepsilon^{-1} = V_0(\varepsilon)$$
$$\tilde{\theta}(V, \varepsilon) = \theta(V, \varepsilon) \qquad V > V_0(\varepsilon)$$

so that, by the properties of $\beta^{[-1]}$, $\theta_0(V_0(\varepsilon), \sigma) = \theta(V_0(\varepsilon), \varepsilon)$. As $\varepsilon \downarrow 0$, $\tilde{\theta}(V, \varepsilon) \uparrow \theta_0(V, \sigma)$ and $\tilde{I}_\varepsilon \to I_\sigma$ in the strong resolvent sense. $\tilde{I}_\varepsilon$ is the collision operator associated to $\tilde{\theta}(V, \varepsilon)$. We have $\tilde{I}_\varepsilon = -\tilde{v}_\varepsilon + \tilde{K}_\varepsilon$ with $\tilde{K}_\varepsilon$ compact and

$$\tilde{v}_\varepsilon(v) = \pi\sigma^2 \int_{V < V_0(\varepsilon)} |\mathbf{v} - \mathbf{v}_1| \omega(v_1)\, d^3v_1 + \pi\sigma^2 V_0(\varepsilon) \int_{V > V_0(\varepsilon)} \omega(v_1)\, d^3v_1 .$$

Min $\tilde{v}_\varepsilon(v) = \tilde{v}_\varepsilon(0) \to v_\sigma(0)$ as $\varepsilon \downarrow 0$ since $V_0(\varepsilon) \to \infty$. Since $I_\sigma \leq \tilde{I}_\varepsilon$ we see that the spectrum of I_σ in the gap $(-v_\sigma(0), 0]$ is purely discrete, despite the fact that K_σ is not relatively compact with respect to $v_\sigma(v)$. As $\sigma \to \infty$ we would recover our limiting operator I_0 again. $\sigma_{\text{ess}}(I_\sigma) = (-\infty, -v_\sigma(0)]$ because we can find a sequence f_n, $\|f_n\| = 1$, with $\|(I_\sigma - \lambda)f_n\| \to 0$ for λ in this interval. For f_n we can choose

$$f_n(\mathbf{v}) = \left(\frac{4\pi}{3}\right)^{-1/2} n^{3/2}, \quad |\mathbf{v} - \mathbf{v}_0| < \frac{1}{n}$$
$$f_n(v) = 0$$

$\mathbf{v}_0$ is any vector with $v(v_0) = -\lambda$. f_n converges only weakly to 0, but by means of Drange's estimates we can show that $\|K_\sigma f_n\| \to 0$ strongly, although K_σ is not compact. Since the functions f_n have compact support it is sufficient that $K_\sigma P_R$ is compact, where P_R projects on the functions with support in some ball of radius R. Obviously, $\|(v_\sigma - \lambda)f_n\| \to 0$, $n \to \infty$, and therefore $\|(I_\sigma - \lambda)f_n\| \to 0$.

Note added in proof

Professor Y. Pao kindly informed me that the selfadjointness can be proved rigorously by the Pseudo-diff. operator formalism. The proof is simple, consisting of applications of inequalities for the symbols of the Pseudo-diff. operators as given by Pao. One proves that Ran $(I_l - N) \restriction \mathscr{S}_l$ is dense.

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