A non-homogeneous string model for hadrons

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A non-homogeneous string model for hadrons[1])

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Abstract. By means of a non-uniform interaction among partons we have introduced a variable elasticity and density of mass into the Nambu–Susskind string. These functions characterize the string. There is a very general class of strings, related by reparametrizations, which are equivalent to the homogeneous dual string model. Further we generalize Veneziano amplitudes. States are much less degenerated than in conventional dual models. Four and five-point amplitudes are explicitly computed showing correct Regge behaviour.

1. Introduction

The so-called string model of hadrons was first introduced independently by Nambu [1] and Susskind [2] defining boson covariant fields $\phi_\mu(\theta)$, and their canonical conjugate $\pi_\mu(\theta)$, as Fourier expansions of annihilation, $a_\mu^{(l)-}$, and creation, $a_\mu^{(l)+}$, operators of four-dimensional harmonic oscillators, with level spacing equal to l, and canonical commutation relations

$$[a_\mu^{(l)-}, a_\nu^{(n)+}] = \delta_{\mu\nu}\,\delta_{en}. \tag{1.1}$$

Nevertheless if we imagine that hadrons are built up of an indeterminate number of point-like constituents, partons [3], we can get a deeper interpretation of the string model making a dynamical hypothesis for parton interactions. We can suppose the hadron boosted with a very large hyperbolic angle, ω, in such a way that the Einstein time dilation effect slows, for some observer at rest, the internal motion as much as we want. This is the impulse approximation, useful because the light can see in this frame the structure of hadrons. If the boost is performed along z-direction one can extract from the infinite z-component of momentum K_z a longitudinal finite fraction $\eta = \sqrt{2}\,K_z\,e^{-\omega}$, where the divergence has been removed. One can prove that in the infinite-momentum-frame the motion of partons is Galilean in the transverse plane [4], η playing the role of mass and $m^2/2\eta$ the role of binding energy. The kinetic energy of the ith parton is

$$E_i = \frac{K_i^2 + m^2}{2\eta_i} \tag{1.2}$$

where K_i is the transverse momentum and m the mass of the parton.

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Further we shall use a multiperipheral symmetrized picture in which the cloud of partons is generated by a bare hadron, with longitudinal fraction unity, which cascaded in the remote past into two bare partons – each one of these two partons cascades into two other more and so on. We can characterize the partons by an angular variable θ given as a function of the longitudinal momentum, $\eta(\theta) = \lambda_0/\pi \sin\theta$ ($0 \leq \theta \leq \pi$), where the constant $\lambda_0 = \pi\eta_{\max}$ is given by the value at which νW_2 begins to decrease sharply to zero ($\eta_{\max} \simeq \frac{1}{3}$). It is supposed that the flow of longitudinal fraction is strongly damped up and down the chain. As a result of this model partons random walk in the transverse plane. The dynamical hypothesis underlying the string parton model is given by a harmonic potential for the interaction between nearest-neighbour partons. Thence the potential energy between partons i and $i+1$ is given by

$$v_{i,i+1} = (g/2\eta_i)(x_i - x_{i+1})^2 \tag{1.3}$$

where X_i is the transverse coordinate of the ith parton, a function of θ_i and the proper time τ.

In a scattering process the hypothesis of nearest-neighbour interaction implies that only partons having longitudinal fraction $(k_z)^{-1}$, wee partons, are responsible for strong interactions, the distribution of partons along the chain being $dN/d\theta = (\lambda_0 \sin\theta)^{-1}$, in accordance with Feynman's distribution law [3] for the multiplicity of fragments $dN_{k_z\to\infty} \sim dk_z/k_z$. Using the Galilean analogy, $k_i = \eta_i(d/d\tau)X_i$, and the continuum approximation $\sum_i \to \int_0^\pi d\theta$, the energy of the system of partons will become

$$H = \frac{1}{2\pi}\int_0^\pi \left[G\left(\frac{\partial X}{\partial\theta}\right)^2 + \left(\frac{\partial X}{\partial\tau}\right)^2\right] d\theta, \tag{1.4}$$

G being a constant with dimensionality of (mass)^2. From (1.4) the equation of motion is found to be

$$\frac{\partial^2 X}{\partial\tau^2} = G\frac{\partial^2 X}{\partial\theta^2} \tag{1.5}$$

If we define the density of transverse momentum as a vector in the (τ, θ)-plane

$$\mathscr{P}_\tau(\theta) = \frac{1}{\pi}\frac{\partial X}{\partial\tau} \qquad \mathscr{P}_\theta(\theta) = -\frac{G}{\pi}\frac{\partial X}{\partial\theta} \tag{1.6}$$

equation (1.5) is the continuity equation describing momentum conservation. The hadron behaves in this model as a homogeneous string with free ends because the boundary conditions are $\mathscr{P}_\theta(\theta = 0, \pi) = 0$, in order to avoid a flux across the ends of the string. The solution of (1.5) gives the well known decomposition of X into modes of oscillation

$$X(\tau, \theta) = X_{\text{c.m.}} + K\tau - i\sqrt{2}\sum_{l=1}^{\infty} e^{-1/2}\{a_e^+ e^{i\sqrt{G}\,l\tau} - a_l^- e^{-i\sqrt{G}\,l\tau}\}\cos l\theta \tag{1.7}$$

where K is the total transverse momentum. By substitution of (1.7) into (1.4) we can express the energy in an operational form

$$H = \tfrac{1}{2}(K^2 + M_0^2) + 2G\sum_{l=1}^{\infty} e a_l^+ a_l^- \tag{1.8}$$

which gives the Veneziano spectrum provided that $G = \frac{1}{4}$. Being wee partons ($\theta = 0, \pi$) responsible of strong interactions, we define the vertex function, as usually, by

$$V(Q) = :e^{iQX(0)}: + :e^{iQX(\pi)}: \tag{1.9}$$

where normal ordering is introduced to substract infinities as in ordinary field theories.

The four point amplitude

$$A(Q_1, Q_2) = (0 \mid V(Q_1)(s - M^2)^{-1} V(Q_2) \mid 0) \tag{1.10}$$

is computed using (1.8,9) and the result is the Veneziano amplitude, or beta-function, if the mass of the string ground state is fixed to the value $M_0^2 = -1$.

The rest of the paper is organized as follows:

In Section 2 the consequences of introducing a non-homogeneity in the string are investigated. In fact we can suppose, in the context of nearest-neighbour interactions, that the coupling constant in (1.3) can be in general i-dependent. Thus we can introduce a non-uniform interaction. It is equivalent, from equation of motion, to consider a string whose modulus of elasticity is not a constant, but rather a function of the longitudinal coordinate of the string. To take into account a non-uniformity in the mass distribution we must introduce another multiplicative factor, parton dependent, in the expression of partons kinetic energy, equation (1.2). It is interesting to note that there is a very general class of non-homogeneous strings having the same Hamiltonian, but different Hamiltonian density, and thus the same spectrum of states as the Veneziano model. The differential equation satisfied by a general non-homogeneous string is called Sturm–Liouville eigenvalue problem.

In Section 3 a particular class of non-homogeneous models, called (p, q)-models, is studied. The relevance of such models is that their differential equation of motion is exactly solvable and they are able to generalize the Veneziano model. Their energy is computed, becoming τ-independent thanks to the orthogonality properties of Jacobi polynomials. For the case $p = 0$, $q = \frac{1}{2}$ one finds the energy of dual resonance models.

In Section 4 we study the $(0, \frac{1}{2}\text{-}\varepsilon)$-model, ε being a small parameter, as a generalization of the Veneziano $(0, \frac{1}{2})$-model. All functions are expanded up to first order in ε. To do that we need to approach Veneziano in a different way for each different mode of excitation. It is indeed possible because the energy is simply the sum of the energies generated by different modes, due to the orthogonality of transverse functions corresponding to different modes. As usual the total coordinate is defined as the sum over the modes, including the zero one. What we really do is to associate a different string to each mode. In this way we can compute the energy. The states (eigenstates of the Hamiltonian) are the same as in dual resonance models, but deplaced in the J–s plane in such a way that they become less degenerated. States of small mass are not degenerated at all. The first degenerated state appears at a mass^2 greater than 3 (GeV)^2. This degeneracy increases with the energy and at a very high energy one finds the exponential behaviour predicted by the statistical model.

The four-point amplitude is computed in this model. In the s-channel the amplitude shows the contribution of poles corresponding to eigenstates of the Hamiltonian. In the s and the t-channel the amplitude has Regge behaviour corresponding to the exchange of usual linear Regge trajectories in crossed channels.

The five-point amplitude is evaluated and its asymptotic behaviour investigated. It is proved to reggeize in simple-Regge, double-Regge and helicity-asymptotic limits as in combined Regge and helicity limit. The analytic continuation of the residue of Regge-poles is performed in each case.

2. The Sturm–Liouville problem

We shall now suppose a non-uniform interaction between nearest-neighbour partons and also a non-uniform longitudinal mass by means of two functions depending on i. Because of the one-to-one correspondence between i and θ, these functions must be dependent on θ_i. In this way the generalized dynamical hypothesis can be written as

$$\tilde{E}_i = \rho(\theta_i)E_i \qquad \tilde{V}_{i,i+1} = \sigma(\theta_i)V_{i,i+1} \tag{2.1}$$

where E_i and $V_{i,i+1}$ are defined in (1.2, 3). That is equivalent to considering a non-homogeneous string model for hadrons, where $\sigma(\theta)$ denotes the modulus of elasticity and $\rho(\theta)$ the mass per unit length of the string. The Hamiltonian is computed in the same way as in Section 1, and it is given by

$$H = \frac{1}{2\pi}\int_0^\pi \left[G\sigma(\theta)\left(\frac{\partial X}{\partial\theta}\right)^2 + \rho(\theta)\left(\frac{\partial X}{\partial\tau}\right)^2\right] d\theta \tag{2.2}$$

and the equation of motion

$$\frac{1}{G}\rho(\theta)\frac{\partial^2 X}{\partial\tau^2} = \frac{\partial}{\partial\theta}\left\{\sigma(\theta)\frac{\partial X}{\partial\theta}\right\}. \tag{2.3}$$

Trying to find a factorized solution

$$X(\tau, \theta) = A(\theta)B(\tau) \tag{2.4}$$

we get

$$\frac{d^2}{d\tau^2}B(\tau) + \lambda B(\tau) = 0 \tag{2.5}$$

and

$$\frac{d}{d\theta}\left[\sigma(\theta)\frac{dA}{d\theta}\right] + \frac{\lambda}{G}\rho(\theta)A(\theta) = 0. \tag{2.6}$$

We are faced with the problem of determining the 'eigenvalues' of equation (2.6) for which a non-trivial solution exists. This is called the Sturm–Liouville eigenvalue problem [5].

Before analyzing equation (2.6) we shall investigate the conditions that functions $\sigma(\theta)$ and $\rho(\theta)$ must satisfy so that by a suitable change of variables $y = G(\theta)$, equation (2.2) becomes equation (1.4).

The answer is straightforward and the first condition is

$$\sigma(\theta)\rho(\theta) = 1 \tag{2.7}$$

the function G being fixed by

$$G(\theta) = \pm \int \rho(\theta)\, d\theta \tag{2.8}$$

and the second condition is, obviously

$$G(0) = 0 \qquad G(\pi) = \pi. \tag{2.9}$$

Thence the solution to equation (2.4) can be written, in this case, as

$$X(\tau, \theta) = X_0(\tau) - i\sqrt{2} \sum_{l=1}^{\infty} l^{-1/2}\{a_e^+ e^{i\sqrt{G}l\tau} - a_l^- e^{-i\sqrt{G}l\tau}\} \cos(lG(\theta)) \tag{2.10}$$

and equation (2.9) insures that transverse conditions for wee partons are the same in all these models as in the homogeneous string configuration. From (2.10) one gets the same operational form for the energy as in the Veneziano model, given by equation (1.4).

As an example of non homogeneous dual string model we define the $(0, \frac{1}{2})$-model by the inhomogeneity functions

$$\rho(\theta) = \sigma^{-1}(\theta) = \{\theta(\pi - \theta)\}^{-1/2} \tag{2.11}$$

satisfying conditions (2.7–9), and the function $G(\theta)$ is given by

$$G(\theta) = \arccos(1 - 2\theta/\pi). \tag{2.12}$$

The Hamiltonian coming from this model can be expressed by (1.8), so it shows the Veneziano spectrum, and the vertex function $:\exp(iQX(0)): + :\exp(iQX(\pi)):$ is the same as in conventional dual models. Hence the four-point amplitude (1.10) is the beta function.

Up to here we have learned that duality is not a special privilege of the homogeneous string. Furthermore one can prove that all dual models are related by a suitable reparametrization in the formalism of Goddard, Goldstone, Rebbi and Thorn [6].

3. The (p, q)-model

We shall generalize the $(0, \frac{1}{2})$-Veneziano model by means of a more general definition of dynamical hypothesis of equation (2.1).

The (p, q)-model is characterized by the following inhomogeneity functions

$$\sigma(\theta) = \theta^q(\pi - \theta)^{p-q+1} \qquad \rho(\theta) = \theta^{q-1}(\pi - \theta)^{p-q}. \tag{3.1}$$

Equation (2.6) with definitions (3.1) and additional conditions

$$q > 0 \qquad p - q > -1 \tag{3.2}$$

is a Sturm–Liouville differential equation whose solution is given by the lth Jacobi polynomial $G_l(p, q, \theta/\pi)$ [5] corresponding to the eigenvalue

$$\lambda = Gl(p + l) \tag{3.3}$$

with the boundary condition that the solution remains finite for $\theta = 0, \pi$. The solution of equation (2.5) is, in this way

$$B_l(\tau) = -i\sqrt{2}\,\{l(p + l)\}^{-1/4}\{a^+(l, \tau) - a^-(l, \tau)\} \tag{3.4}$$

with

$$a^{\pm}(l, \tau) = \exp\{\pm i[Gl(p + l)]^{1/2}\tau\}a_l^{\pm}. \tag{3.5}$$

Defining the total solution of equation (2.4) as a summation over modes, we have

$$X(\tau, \theta) = X_0(\tau) - i\sqrt{2}\sum_{l=1}^{\infty}\{l(p + l)\}^{-1/4}\{a^+(l, \tau) - a^-(l, \tau)\}G_l(p, q, \theta/\pi). \tag{3.6}$$

The equation of motion is again equivalent to the continuity equation of the bivector describing the internal density of momentum

$$\begin{aligned} P_\tau(p, q, \theta) &= \frac{1}{\pi}\theta^{q-1}(\pi - \theta)^{p-q}\frac{\partial X}{\partial \tau} \\ P_\theta(p, q, \theta) &= -\frac{G}{\pi}\theta^q(\pi - \theta)^{p-q+1}\frac{\partial X}{\partial \tau} \end{aligned} \tag{3.7}$$

where physical boundary conditions $P_\theta(\theta = 0, \pi) = 0$ are automatically satisfied from (3.2).

By substitution of (3.1–6) into (2.2), and using the orthogonality properties of Jacobi polynomials [5], one can find the energy of (p, q)-model as

$$H(p, q) = \pi^{p-1}\left\{B(p - q + 1, q)\frac{K^2}{2} + 4G\sum_{l=1}^{\infty}A_l(p, q)a_l^+a_l^-\right\} + \frac{M_0^2}{2} \tag{3.8}$$

where M_0 is the energy of the string ground state and

$$A_l(p, q) = \{l(p + l)\}^{1/2}\frac{l!\,\Gamma^2(q)\Gamma(l + p - q + 1)}{(2l + p)\Gamma(l + p)\Gamma(l + q)}. \tag{3.9}$$

One can easily see that in the case $p = 0$, $q = \frac{1}{2}$ we have in (3.8, 9)

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = \pi \qquad A_l(0, \tfrac{1}{2}) = \frac{\pi}{2}l \tag{3.10}$$

and the value of $H(0, \frac{1}{2})$ agrees with that of the homogeneous dual string model as given by equation (1.8).

4. A generalized Veneziano model

In this Section we want to describe a model generalizing the Veneziano $(0, \frac{1}{2})$-model by means of a small parameter, $0 \leq \varepsilon \leq 1$, introduced in such a way that the behaviour of the dual string model would be obtained in the limit $\varepsilon \to 0$.

We can fix $p = 0$, $q = \frac{1}{2} - \varepsilon$ in Section 3 and expand each function around $\varepsilon = 0$. Doing that we find, for instance, that the expansion $\Gamma(l + \frac{1}{2} - \varepsilon) = \Gamma(l + \frac{1}{2})[1 - \varepsilon\psi(l + \frac{1}{2})] + 0(\varepsilon^2)$ has no sense because $\psi(l + \frac{1}{2}) \sim \log l$ as $l \to \infty$ and we can always find an integer ε-dependent, $N(\varepsilon)$, such that if $l > N(\varepsilon)$, $\varepsilon\psi(l + \frac{1}{2}) > 1$. This suggests to us that if we want to approach the Veneziano model by means of some small parameter ε, it must be dependent on the particular mode of excitation considered, $\varepsilon(l)$, and not a constant. This would give a slightly different interpretation of the hadron in terms of the modes of oscillation of a string. Up to here all hadron excitations have been associated with the modes of a string. In fact the solution to equation (2.3–6) is given by $X_l(\tau, \theta) = B_l(\tau)G_l(p, q, \theta/\pi)$ with any l going from zero (if zero mode is included) to infinity. Equation of motion is not depending on l, so we can take as general solution X the summation over X_l. So different modes are associated to the same differential equation. If p and q are functions of l, $p = p(l)$ and $q = q(l)$, there will be a different differential equation for each mode l. Taking as particular solution $X_l(\tau, \theta) = B_l(\tau)G_l(p(l), q(l), \theta/\pi)$ and as total solution $X = \sum_l X_l$, all equations of Section 3 remain valid. In this way we shall define the $(0, \frac{1}{2}\text{-}\varepsilon)$-model by means of the following prescription

$$\begin{aligned} &p(l) = 0 && l \geq 0 \\ &q(0) = q(1) = \tfrac{1}{2} \qquad q(l) = \tfrac{1}{2} - \varepsilon(l) && l \geq 2 \end{aligned} \tag{4.1}$$

where $\varepsilon(l) = \{4[\psi(l + \frac{1}{2}) - \psi(\frac{1}{2})]\}^{-1}\varepsilon$. The parameter ε is fixed and less than one, and the factor dependent on l is particularly useful to simplify the formulas, decreasing logarithmically when l grows to infinity.

Expanding the function (3.9) up to first order in ε one finds

$$H(0, \tfrac{1}{2} - \varepsilon) = \tfrac{1}{2}(K^2 + M^2) \tag{4.2}$$

with

$$M^2 = 4G\{a_1^+a_1^- + \sum_{l=2}^{\infty} l(1 + \varepsilon)a_l^+a_l^-\} + M_0^2. \tag{4.3}$$

Because $p(0) = 0$ and $q(0) = \frac{1}{2}$, the external energy is unchanged with respect to that of the $(0, \frac{1}{2})$-model and fixing $\varepsilon(1) = 0$ the intercept of the leading trajectory is equal to one. This is important because it is the only case where the ghost-killing mechanism of Virasoro operators has been proved [7] to cancel the ghosts.

4.1. The spectrum of states

Taking in (4.2, 3) as coupling constant $G = \frac{1}{4}$ and the mass of the string ground state $M_0^2 = -1$, a taquion, we can compute the mass of intermediate states of the theory

$$|l_1, l_2, \ldots) = (l_1!\, l_2! \ldots)^{-1/2}(a_1^+)^{l_1}(a_2^+)^{l_2}\ldots\,|0) \tag{4.4}$$

and we find they lie, in the J–M^2 plane, on a set of straight line trajectories, the leading one and its daughters (as in dual models) plus a set of parallel trajectories shifted from the formers by an amount equal to $2\varepsilon, 3\varepsilon, \ldots$ The states are much less degenerated and trajectories are now split. Only the leading trajectory has no splitting, reflecting the fact that its states are the only non-degenerated in dual models. In the limit $\varepsilon \to 0$ all the new trajectories come back to the old ones and states become again degenerated. In Figure 1 we have shown the states (4.4) in the J–M^2 plane. We can see that up to $M^2 = 3(\text{GeV})^2$ the states are all non-degenerated. The first degenerated state we find is $|0, 0, 0, 1, 0, \ldots)$ with a mass^2 equal to $3 + 4\varepsilon$, coinciding with the first daughter of the state $|0, 2, 0, \ldots)$. The degeneracy is much smaller than in conventional dual spectrum, and to compute its limit at very high energy we can proceed as Fubini, Gordon and Veneziano [8] and define the partition function of the system by the usual formula

$$Z(T) = Tr\{\exp(-H/T)\} \tag{4.5}$$

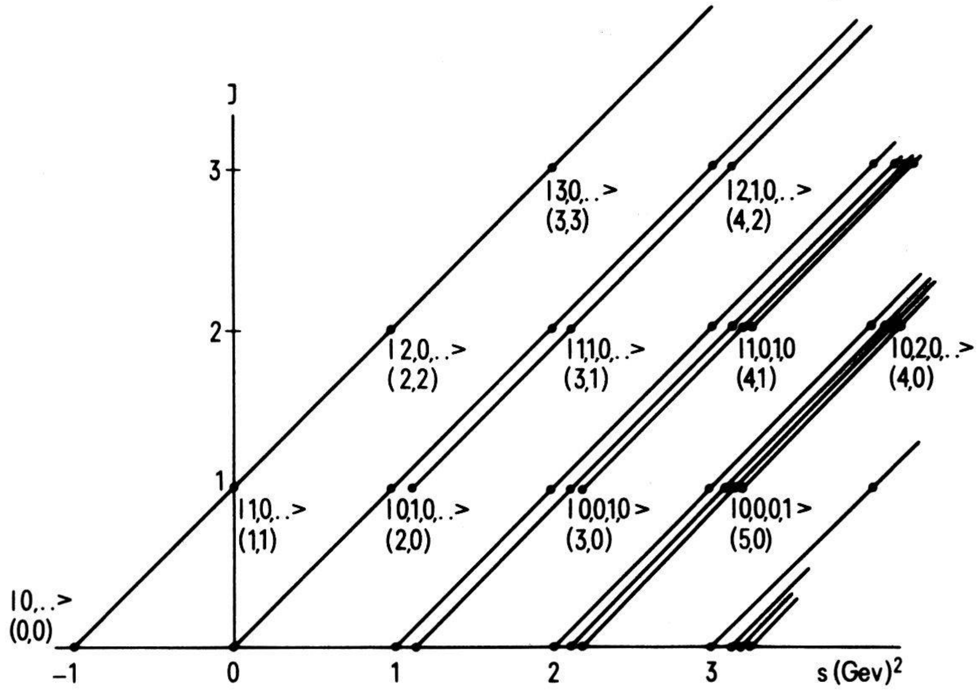

Figure 1
The spectrum of the (0, $\frac{1}{2}$-ε)-model.

where H is given by (4.2, 3). We know from thermodynamics and statistical physics expressions for Gibbs potential

$$G(T) = -T\log Z \tag{4.6}$$

entropy

$$S = -\frac{\partial G}{\partial T} \tag{4.7}$$

and internal energy

$$E = T^2\frac{\partial \log Z}{\partial T} \tag{4.8}$$

Because the level density, or degeneracy of states, is given by the exponential of the entropy, by direct application of (4.2–8) we find

$$D(E) \underset{E\to\infty}{\sim} \exp\left\{\frac{2\pi}{(1+\varepsilon)^2}\left(\frac{2E}{3}\right)^{1/2}\right\} \tag{4.9}$$

Equation (4.9) agrees with the behaviour of level degeneracy in Hagedorn's thermodynamical model [9] and gives the asymptotic behaviour [8] of dual resonance model at $\varepsilon = 0$.

4.2. *The four-point amplitude*

We shall compute now the four point amplitude corresponding to the multiperipheral configuration of Figure 2, as given by

$$A_4 = (0, p_0 \mid V_\varepsilon(p_1)(s - M^2)^{-1} V_\varepsilon(p_2) \mid 0, p_3) \tag{4.10}$$

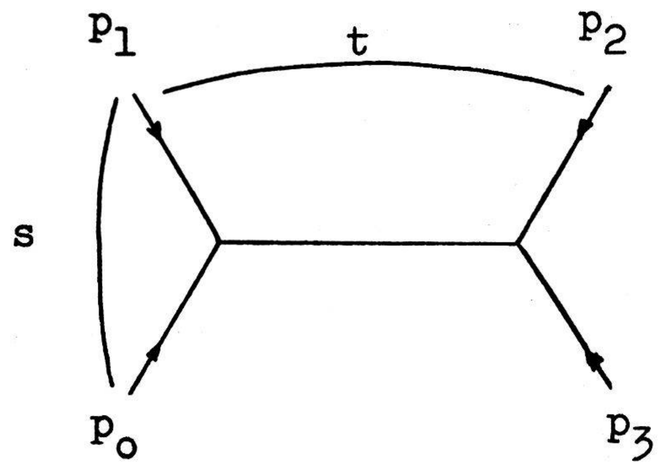

Figure 2
The four-point amplitude.

where the kinematical invariants of the reaction are defined by

$$s = -(p_0 + p_1)^2 \qquad t = -(p_1 + p_2)^2 \tag{4.11}$$

and the vertex

$$V_\varepsilon(Q) = :e^{iQX_\varepsilon(0)}: + :e^{iQX_\varepsilon(\pi)}: = :e^{iQX(0)}: + :e^{iQX(\pi)}: + 0(\varepsilon) \tag{4.12}$$

X_ε given by (3.6) and X by (1.7).

We shall take for the vertex a zero-th order approximation in the parameter ε. We have in this way the same vertex as in the Veneziano $(0, \frac{1}{2})$-model. For the propagator in (4.10) we take the usual integral representation

$$(s - M^2)^{-1} = \int_0^1 Xs^{-M^2-1}\, dx. \tag{4.13}$$

Using the canonical commutation relations (1.1) and the definition of coherent states as the action over the vacuum of the exponential [10]

$$|f_e) = \exp(f_e a_e^+)\,|0) \tag{4.14}$$

with the normalization

$$(f_m \mid g_n) = \exp(f_m^* g_n)\delta_{mn} \tag{4.15}$$

we get for the amplitude (4.10) the following expression

$$A_4(s, t) \equiv A_4(\alpha(s), \alpha(t)) = \int_0^1 dx\, x^{-\alpha(s)-1}(1 - x^{1+\varepsilon})^{-\alpha(t)-1}$$
$$\exp\{-(\alpha(t) + 1)(x^{1+\varepsilon} - x)\} \tag{4.16}$$

where $\alpha(x) = x + 1$ is the linear Veneziano–Regge trajectory. Obviously in the limit $\varepsilon \to 0$ we regain the Veneziano amplitude.

Requirement of Regge asymptotic behaviour is a very important test, both from phenomenological and theoretical point of view, for a function to describe a scattering amplitude. In fact Regge behaviour is deduced from very basic principles of S-matrix theory, as first and second kind analyticity. In the following we shall compute the asymptotic behaviour of $A_4(s, t)$ in the two independent channels.

4.2.1. t-channel Regge limit. In this Section we shall compute the limit of equation (4.16) as $|t| \to \infty$. Since (4.16) is only defined for negative α, we first take the limit $t \to -\infty$ and then continue in s.

We shall proceed as Bardakçi and Ruegg [11], using a generalized Laplace's method, making the change of variables $x = y/(-t)$, and expanding the integrand of (4.16) in a power series of $(t)^{-1}$. Using

$$\left[1 - \frac{y^{1+\varepsilon}}{(-t)^{1+\varepsilon}}\right]^{-t-2} \exp\left[(t+2)\left(\frac{y}{t} - \frac{y^{1+\varepsilon}}{(-t)^{1+\varepsilon}}\right)\right] \underset{t \to -\infty}{\sim} \exp[-y] \tag{4.17}$$

we obtain

$$A_4(s, t) \underset{t \to -\infty}{\sim} (-t)^{\alpha(s)} \int_0^\infty dy\, y^{-\alpha(s)-1} e^{-y} \tag{4.18}$$

that can be written as

$$A_4(s, t) \underset{t \to -\infty}{\sim} (-t)^{\alpha(s)} \Gamma(-\alpha(s)) \tag{4.19}$$

which shows poles at $\alpha(s) = 0, 1, 2, \ldots$

The behaviour (4.19) corresponds to the exchange, in the s-channel, of the Regge trajectory $\alpha(s)$.

4.2.2. s-channel Regge limit. In order to compute the limit $s \to -\infty$ of equation (4.16) it is convenient to first make the transformation of variables $x \to 1 - x$ so that it can be written as

$$A_4(s, t) = \int_0^1 dx(1-x)^{-\alpha(s)-1}[1-(1-x)^{1+\varepsilon}]^{-\alpha(t)-1} \exp\{(\alpha(t)+1)$$
$$[1 - x - (1-x)^{1+\varepsilon}]\}. \tag{4.20}$$

Using the change $x = y/(-s)$ and expanding the integrand as

$$(1 + y/s)^{-s-2} \underset{s \to \infty}{\sim} e^{-y} \tag{4.21}$$

and

$$[1 - (1 + y/s)^{1+\varepsilon}]^{-t-2} \exp\{(t+2)(1 + y/s - (1 + y/s)^{1+\varepsilon})\} \sim$$
$$\left(\frac{-s}{1+\varepsilon}\right)^{t+2} y^{-t-2} \tag{4.22}$$

we get

$$A_4(s, t) \underset{s \to -\infty}{\sim} (1+\varepsilon)^{-\alpha(t)-1}(-s)^{\alpha(t)}\, \Gamma(-\alpha(t)) \tag{4.23}$$

corresponding to the exchange of the Regge trajectory $\alpha(t)$ in the crossed channel.

4.3. The five-point amplitude

In this Section we shall study the five-point amplitude corresponding to the configuration of Figure 3, as given by

$$A_5 = (p_0, 0 \mid V_\varepsilon(p_1)(s_1 - M^2)^{-1} V_\varepsilon(p_2)(s_2 - M^2)^{-1} V_\varepsilon(p_3) \mid p_4, 0) \tag{4.24}$$

where the kinematical invariant of the reaction are defined as

$$s_i = -\left(\sum_{j=0}^{i} p_j\right)^2 \qquad \tau_i = -(p_i + p_{i+1})^2 \qquad \tau = -(p_0 + p_4)^2. \tag{4.25}$$

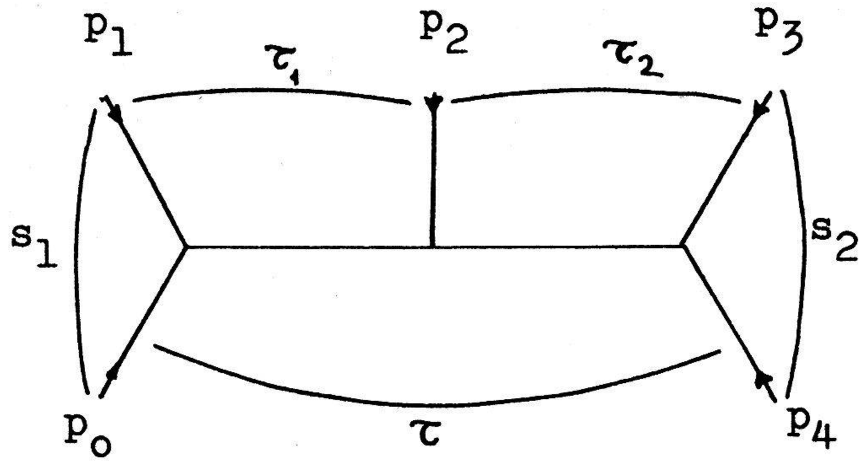

Figure 3
The five-point amplitude.

Using the same values of $V_\varepsilon(p)$ and M^2 as in Section 4.2 it is straight-forward to get for the amplitude the integral representation

$$A_5(s_1, s_2, \tau_1, \tau_2, \tau) = \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{-\alpha(s_1)-1} x_2^{-\alpha(s_2)-1} \tag{4.26}$$

$$(1 - x_1^{1+\varepsilon})^{-\alpha(\tau_1)-1}(1 - x_2^{1+\varepsilon})^{-\alpha(\tau_2)-1}(1 - x_1^{1+\varepsilon}x_2^{1+\varepsilon})^{-\alpha(\tau)+\alpha(\tau_1)+\alpha(\tau_2)}$$
$$\exp\{-(\alpha(\tau_1) + 1)(x_1^{1+\varepsilon} - x_1) - (\alpha(\tau_2) + 1)(x_2^{1+\varepsilon} - x_2) -$$
$$(\alpha(\tau) - \alpha(\tau_1) - \alpha(\tau_2))((x_1x_2)^{1+\varepsilon} - x_1x_2)\}$$

which is a generalization of the Bardakçi–Ruegg [11] five-point formula.

In order to compute the asymptotic behaviour of (4.26) we shall use Bardakçi–Ruegg's method [11], as in Section 4.2, and follow closely the work of Brower, De Tar and Weis [12] to analyze the asymptotic limits of five-point amplitude.

4.3.1. Simple-Regge limit. The limit $\tau_1, \tau \to \infty$, s_1, s_2, τ/τ_1 fixed is usually called [12] simple-Regge limit of five-point amplitude. Since (4.26) is defined for negative α, we first take $\tau_1, \tau \to -\infty$ and then continue in τ_1, τ. The region $x_1 \approx 0$ dominates the integral in this limit. Let us perform, in (4.26), the transformation of variables

$x_1 = y_1/(-\tau_1)$ and expand the integrand in a power series of $(\tau_1)^{-1}$. After a lengthy but straightforward calculation we get

$$A_5 \underset{\substack{\tau_1, \tau \to -\infty \\ \tau_2, \tau/\tau_1, s_1, s_2 \,\text{fixed}}}{\sim} (-\tau_1)^{\alpha(s_1)}\, g(s_1, s_2, \tau_2, \tau/\tau_1, \varepsilon) \tag{4.27}$$

which corresponds to the asymptotic behaviour of a diagram where a Regge pole is exchanged in the s_1-channel. The residue of the Regge pole is given by the function

$$\begin{gathered} g = \int_0^\infty dy_1 \int_0^1 dx_2\, y_1^{-\alpha(s_1)-1}\, x_2^{-\alpha(s_2)-1}\, (1 - x_2^{1+\varepsilon})^{-\alpha(\tau_2)-1} \\ \exp\{-(\alpha(\tau_2) + 1)(x_2^{1+\varepsilon} - x_2) - y_1 + y_1 x_2 - y_1 x_2 \tau/\tau_1\} \end{gathered} \tag{4.28}$$

or using the simple change of variables $(1 - x_2)y_1 \to y_1$ we get

$$\begin{gathered} g = \int_0^\infty dy_1 \int_0^1 dx_2\, y_1^{-\alpha(s_1)-1}\, x_2^{-\alpha(s_2)-1}(1 - x_2)^{\alpha(s_1)}(1 - x_2^{1+\varepsilon})^{-\alpha(\tau_2)-1} \\ \exp\{-(\alpha(\tau_2) + 1)(x_2^{1+\varepsilon} - x_2)\} \exp\left\{-y_1 - y_1 \frac{x_2}{1 - x_2}\frac{\tau}{\tau_1}\right\}. \end{gathered} \tag{4.29}$$

The integral (4.29) only converges for $\tau/\tau_1 \geq 0$. It is therefore not an entire function of τ/τ_1. To obtain the analytic continuation of (4.29) we can use the identity

$$\exp(Z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\lambda\, \Gamma(-\lambda)(-Z)^\lambda. \tag{4.30}$$

The y_1 integral in (4.29) can be performed with the result

$$g = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\lambda\, \Gamma(\lambda - \alpha(s_1))\, \Gamma(-\lambda)\tilde{g}(\lambda, s_1, s_2, \tau_2, \varepsilon)(\tau/\tau_1)^\lambda \tag{4.31}$$

where

$$\begin{gathered} \tilde{g} = \int_0^1 dx_2\, x_2^{\lambda-\alpha(s_2)-1}(1 - x_2)^{-\lambda+\alpha(s_1)}(1 - x_2^{1+\varepsilon})^{-\alpha(\tau_2)-1} \\ \exp\{-(\alpha(\tau_2) + 1)(x_2^{1+\varepsilon} - x_2)\}. \end{gathered} \tag{4.32}$$

In particular if $\tau/\tau_1 < 1$ the contour in (4.31) can be closed to the left, picking up poles at $\lambda = \alpha(s_1) - n$ and $\lambda = \alpha(s_2) - m$ $(n, m = 0, 1, \ldots)$.

4.3.2. Double-Regge limit. The limit $\tau_1, \tau_2, \tau \to \infty$, $\tau/(\tau_1\tau_2)$, s_1, s_2 fixed is called the double-Regge limit of the five-point amplitude. It can be obtained by taking the further limit $\tau_2 \to \infty$, $\tau/(\tau_1\tau_2)$ fixed, on (4.28). Thus making the change of variables $x_2 = y_2/(-\tau_2)$ and expanding the integrand we get

$$A_5 \underset{\substack{\tau_1, \tau_2, \tau \to -\infty \\ \tau/\tau_1\tau_2, s_1, s_2 \,\text{fixed}}}{\sim} (-\tau_1)^{\alpha(s_1)}(-\tau_2)^{\alpha(s_2)}\, f(s_1, s_2, \tau/\tau_1\tau_2) \tag{4.33}$$

where

$$f = \int_0^\infty dy_1 \int_0^\infty dy_2 y_1^{-\alpha(s_1)-1} y_2^{-\alpha(s_2)-1} \exp\left\{-y_1 - y_2 + y_1 y_2 \frac{\tau}{\tau_1\tau_2}\right\} \tag{4.34}$$

which corresponds to Regge trajectories exchanged in s_1 and s_2 channels. The residue is only defined for negative $\tau/(\tau_1\tau_2)$. The analytic continuation of (4.34) in the variable $\tau/(\tau_1\tau_2)$ can be found using again (4.30) to perform y_1 and y_2 integrations with the result

$$f = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\lambda\, \Gamma(\lambda - \alpha(s_1))\, \Gamma(\lambda - \alpha(s_2))\, \Gamma(-\lambda)\left(-\frac{\tau}{\tau_1\tau_2}\right)^{\lambda} \tag{4.35}$$

4.3.3. Helicity Asymptotic limit. The limit $\tau \to \infty$, τ_1, τ_2, s_1, s_2, fixed, is called the helicity asymptotic limit [12]. Looking at (4.26) we can see that $\tau \to -\infty$ causes $x_1 \approx 0$ or $x_2 \approx 0$ to dominate in the integral. In fact the integral does not converge if both x_1 and $x_2 \approx 0$. There are, therefore, two contributions. The first one, corresponding to $x_1 \approx 0$, can be extracted making in (4.26) the change $x_1 = y_1/(-\tau)$ and expanding the integrand in powers of $(\tau)^{-1}$, taking the limit $\tau \to -\infty$. To get the contribution corresponding to the region of integration $x_2 \approx 0$ we perform in (4.26) the transformation $x_2 = y_2/(-\tau)$ and expand the integrand as usually. The result can be written as

$$\begin{aligned} A_5 \underset{\substack{\tau\to-\infty \\ s_1, s_2, \tau_1, \tau_2 \text{fixed}}}{\sim} & \Gamma(-\alpha(s_1))(-\tau)^{\alpha(s_1)} A_4(\alpha(s_2) - \alpha(s_1), \alpha(\tau_2)) \\ & + \Gamma(-\alpha(s_2))(-\tau)^{\alpha(s_2)} A_4(\alpha(s_1) - \alpha(s_2), \alpha(\tau_1)) \end{aligned} \tag{4.36}$$

where the function A_4 is defined in (4.16). Behaviour (4.36) is in agreement with the general form [12]. The helicity limit behaviour is important because it means that for negative $\alpha(s_1)$ and $\alpha(s_2)$ we can write an unsubtracted dispersion relation in τ with the other α's in (4.26) fixed.

Finally the combined Regge and helicity limit $\tau_1 \to \infty$ and $\tau/\tau_1 \to \infty$ is easily deduced from (4.36) giving

$$\begin{aligned} A_5 \underset{\tau_1, \tau/\tau_1 \to \infty}{\sim} & \Gamma(-\alpha(s_1))(-\tau)^{\alpha(s_1)} A_4(\alpha(s_2) - \alpha(s_1), \alpha(\tau_2)) \\ & + \Gamma(-\alpha(s_2))(-\tau)^{\alpha(s_2)} \Gamma(\alpha(s_2) - \alpha(s_1))(-\tau_1)^{\alpha(s_1)-\alpha(s_2)} \end{aligned} \tag{4.37}$$

5. Conclusion

We have studied through this paper the consequences of introducing a non-uniform parton interaction and a non-uniform mass distribution by means of two functions which modify the usual dynamical hypothesis giving rise to the so-called conventional dual string model. We are tempted to interpret this as an average effect of non-neighbour partons modifying the pure oscillatory pattern. This is equivalent to be considered for the hadron a non-homogeneous string model.

We have seen that duality is not a special privilege of the homogeneous string. There is a very large set of models having the same spectrum and whose wee partons (ends of the string) have the same transverse coordinate as in the Veneziano model.

It is known that while hadrons are emitted from the edge of the Koba–Nielsen disc [13], wee partons, photons emerge from its interior [14]. So it should be interesting to investigate off-shell currents and scalar amplitudes [15] inside the context of these new distributions of partons in the Nambu–Susskind strip. One of these models, the $(0, \frac{1}{2})$-model, is generalized to the (p, q)-models. In this way we can approach the Veneziano model by variation of the parameters p and q from the values 0 and $\frac{1}{2}$, respectively. We obtain an ε-dependent model which coincides with the Veneziano model at $\varepsilon = 0$. It is a trivial but effective generalization of conventional dual models because it shows splitting of trajectories in the s-channel, in such a way that the states are much less degenerated. The lowest mass states are not degenerated at all. However, the degeneracy of the model respects, in the high energy limit, the behaviour $\exp(C\sqrt{E})$ given by statistical models [9].

The four-point amplitude shows decomposition in poles in s and t-channels, with polynomial residues, and good asymptotic behaviour in the limits s and $t \to \infty$, as expected from first principles and S-matrix theory.

The five-point amplitude Reggeizes both in simple and double-Regge limit as in helicity asymptotic limit or combined Regge and helicity limits.

Let us remark on the curious fact that there appears a hidden mechanism (or symmetry) which assures Reggeization of N-point amplitudes by means of cancellations of leading (exponential) terms which could violate Regge behaviour when the integrand is expanded as an asymptotic series. In this way the amplitudes Reggeize even if they seem to violate, at first view, the correct Regge behaviour.

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