An application of the third order JWKB-approximation method to prove absolute continuity I. The construction

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Abstract. We employ the approximation method of Jeffreys, Wentzel, Kramers and Brillouin to study the continuous spectra of a class of one-dimensional Schrödinger operators. A typical potential in our class is a potential which at infinity is smaller than the one of von Neumann–Wigner. According to our main theorem the interior of the essential spectrum of such a Schrödinger operator is absolutely continuous.

In this first part of this paper we employ the third order JWKB-approximation method to construct a family of approximating operators to such a Schrödinger operator.

1. Introduction

In their classic paper, 'Über merkwürdige diskrete Eigenwerte' von Neumann and Wigner [1] gave an example of a Schrödinger operator whose spectrum consisted of $[0, \infty)$ together with a strictly positive point eigenvalue. The potential of their operator was oscillating near infinity.

In this paper we exhibit a class of potentials which oscillates somewhat slower than the von Neumann–Wigner potential and the corresponding operator has no strictly positive point-eigenvalues. In fact, in this case the part of the operator over the open positive axis is absolutely continuous. At the same time we illustrate how the third order JWKB-approximation method can be applied to prove this property.

In Section 2 first we describe our class of potentials. Then in Theorem 2.1, which is our main result, we state that the parts of the corresponding Schrödinger operators over the open positive axis are absolutely continuous.

In Section 3 we formulate a set of abstract criteria for absolute continuity. This set of criteria is a simplified version of a set formulated elsewhere [8]. As before, the key requirement is the existence of a family of approximating operators in the sense of the technical Definition 3.1.

In Section 4 we formulate a sufficient condition on the resolvent kernels of a family of operators in order that this family of operators approximate a given operator. This is described in specific terms in Lemma 4.1.

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In Section 5 we employ the third order Jeffreys, Wentzel, Kramers, Brillouin approximation method [11, 12, 18] to construct a family of approximating operators to the operator $L(p)$ of Theorem 2.1. First, in Lemma 5.1 we formulate a version of a result of Sibuya [4, 19], concerning JWKB-approximate solutions of the equation,

$$(\mu - L(p))f(\mu) = 0, \quad \text{Re } \mu \in \mathscr{I}.$$

Let $y(\mu)$ denote such a family of approximate solutions. To this family of approximate solutions corresponds a family of potentials, $q(\mu)$, defined by the property,

$$(\mu - L(q(\mu)))y(\mu) = 0.$$

In general, there is no reason to expect that this family of operators, $L(q(\mu))$, approximates the operator $L(p)$ in the sense of Definition 3.1. Therefore we formulate Condition $0(\mathscr{I})$ on the potential p which ensures that this is the case. We do not claim that the potentials p of Theorem 2.1 satisfy this condition. However, we show in Lemma 5.2 that such a potential p admits a decomposition of the form $p = p_1 + p_2$, where p_1 is short range and p_2 satisfies Condition $O(\mathscr{I})$.

It is far from being evident that the family of potentials $q(\mu)$ corresponding to p_2 is such that the family of operators $L(q(\mu))$ approximates the operator $L(p)$. The proof of this fact requires estimates which will be formulated in the second part of this paper.

For the role of absolute continuity in quantum scattering theory we refer to the paper of Amrein–Georgescu [7]. For the role of self-adjointness in quantum theory we refer to the recent book of Piron [20].

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2. Formulation of the result

We start this section by describing a class of potentials. First we say that a potential p is short range if

$$p \in \mathfrak{L}_1(\mathscr{R}^+) \cap \mathfrak{L}_{2,\text{loc}}(\mathscr{R}^+) \tag{2.1}$$

and

$$\lim_{\xi \to \infty} \int_{\xi}^{\xi+1} |p(\eta)|^2 \, d\eta = 0. \tag{2.2}$$

Secondly we introduce some notations. For a given pair of positive constants (β, γ) we define two functions by

$$a(\beta, \gamma)(\xi) = \left(\frac{1}{1+\xi}\right)^{\beta} \sin(1+\xi)^{\gamma}, \quad \xi \in \mathscr{R}^+ \tag{2.3}$$

and

$$b(\beta,\gamma)(\xi) = \left(\frac{1}{1+\xi}\right)^{\beta} \cos(1+\xi)^{\gamma}, \quad \xi \in \mathscr{R}^{+} \tag{2.4}$$

Then we define

$$\mathfrak{L}(\beta,\gamma,\mathscr{R}^{+}) = \text{Span}\,\{a(\beta,\gamma), b(\beta,\gamma)\}. \tag{2.5}$$

Clearly in this definition we can replace $\mathscr{R}^{+}$ by any of its subintervals and we let

$$\mathfrak{L}(\beta,\gamma,\infty) = \bigcup_{\xi_0 \in \mathscr{R}+} \mathfrak{L}(\beta,\gamma,[\xi_0,\infty)). \tag{2.6}$$

From now on we assume that the potential p admits a decomposition of the form

$$p = p_1 + p_2, \tag{2.7}$$

where p_1 is short range and p_2 is long range in the sense that it belongs to a class of the form (2.6). As usual let $(\mathfrak{C}_0^{\infty}(\mathscr{R}^{+})$ denote the class of infinitely differentiable complex valued functions with compact support in $\mathscr{R}^{+}$. For a given α in $\mathscr{R}_1$ and given potential p define,

$$\mathfrak{D}(L(p)) = \{f : f \in \mathfrak{C}_0^{\infty}(\mathscr{R}^{+}), \text{ and } f(0)\cos\alpha - f'(0)\sin\alpha = 0\}. \tag{2.8}$$

Then define the operator $L(p)$ mapping this set into $\mathfrak{L}_2(\mathscr{R}^{+})$ by

$$L(p)f(\xi) = -\hbar^2 f''(\xi) + \mathrm{p}(\xi)f(\xi). \tag{2.9}$$

It is not difficult to show that the Rellich–Kato theorem [16] implies that for real potentials p satisfying assumption (2.2) this operator is essentially self-adjoint on $\mathfrak{D}(L(p))$. At the same time it follows that the domain of the closure of this operator, is independent of p. In fact, denoting this closure by $L(p)$ again, one has

$$\mathfrak{D}(L(p) = \mathfrak{D}(L(p_1)) = \mathfrak{D}(L(p_2)) = \mathfrak{D}(L(0)).$$

The theorem that follows is our main result and it formulates an absolute continuity criterion for a par) of the operator $L(p)$.

Theorem 2.1. *Suppose that the real potential p admits a decomposition of the form* (2.7) *where* p_1 *is short range and to* p_2 *there is a pair of constants* (β,γ) *such that*

$$\alpha > 0, \quad 1 > \gamma > 0, \quad \text{and} \quad \beta + 3(1-\gamma) > 1, \tag{2.10}$$

and

$$p_2 \in \mathfrak{L}(\beta,\gamma,\infty) \cap \mathfrak{L}_{2,\,\mathrm{loc}}(\mathscr{R}^{+}). \tag{2.11}$$

Let $L(p)$ *denote the closure of the operator defined by relations* (2.8) *and* (2.9). *Then the part of* $L(p)$ *over* $\mathscr{R}^{+}$ *is absolutely continuous, that is,*

$$L(p)(\mathscr{R}^{+}) = L(p)(\mathscr{R}^{+})_{\mathrm{ac}}. \tag{2.12}$$

We shall derive this theorem from an abstract theorem to be stated in Section 3.

We conclude this section by adding two remarks on Theorem 2.1. First we remark that some of the inequalities in assumption (2.10) have to be strict. For von Neumann and Wigner [1, 5] have constructed a potential p such that

$$p_2 \in \mathfrak{L}(1,1,\mathscr{R}^{+}), \quad \text{i.e.,} \quad \beta = 1, \quad \gamma = 1$$

and the operator $L(p)$ has a strictly positive point-eigenvalue. Hence conclusion (2.12) does not hold. Secondly we replace assumption (2.10) by the following, more stringent one,

$$\beta > 0, \quad 1 > \gamma > 0 \quad \text{and} \quad \beta + 2(1 - \gamma) > 1.$$

Then we remark that conclusion (2.12) is implied by an extended version of a theorem of Titchmarsh–Neumark–Walter formulated elsewhere [8a].

3. An abstract criterion for absolute continuity

In this section we formulate a simplified version of a criterion for absolute continuity formulated elsewhere [8b].

Let A be a given self-adjoint operator acting on a given abstract Hilbert space $\mathfrak{H}$. We state a lemma which gives a simple sufficient condition for a part of A to be absolutely continuous. To formulate it we need some notations. To a given interval of reals, $\mathscr{I}$, and angle α, we assign two open regions of the complex plane by setting

$$\mathscr{R}_{\pm}(\mathscr{I}) = \{\mu : \operatorname{Re} \mu \in \mathscr{I}^0, 0 < \pm \arg\mu < \alpha\}, \tag{3.1$_\pm$}$$

where $\mathscr{I}^0$ denotes the interior of the interval $\mathscr{I}$. As usual, we denote by $\mathfrak{B}(\mathfrak{H})$ the space of everywhere defined bounded operators on $\mathfrak{H}$. For a possibly unbounded operator T and for μ in $\rho(T)$, the resolvent set of T, we set

$$R(\mu, T) = (\mu I - T)^{-1} \in \mathfrak{B}(\mathfrak{H}). \tag{3.2}$$

Lemma 3.1. *Suppose that to A and to the given compact interval $\mathscr{I}$ there is a dense subset $\mathfrak{C}$ such that for each pair of vectors (f, g) in $\mathfrak{C} \times \mathfrak{C}$*

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} |(R(\mu)f, g) - (R(\bar{\mu})f, g)| < \infty \tag{3.3}$$

Then $A(\mathscr{I})$, the part of A over $\mathscr{I}$, is absolutely continuous.

It was observed elsewhere that this lemma is an elementary consequence of the resolvent loop-integral formula.

For a class of Schrödinger operators it is possible to factorize the resolvent in a manner which allows us to establish the rather general assumptions of Lemma 3.1. To describe such factorizations we make a digression on forms. Accordingly let $\mathfrak{G}$ be an abstract Banach space and [F] a functional on $\mathfrak{G} \times \mathfrak{G}$ which is linear in the first argument and conjugate linear in the second argument, in short, a sesquilinear form. In analogy to the notion of the norm of an operator we define the norm of the form $[F]$ by

$$\|[F]\| = \sup_{f \neq 0, g \neq 0} \frac{|[F](f, g)|}{\|f\|_{\mathfrak{G}} \|g\|_{\mathfrak{G}}} \tag{3.4}$$

and we denote by $\mathfrak{F}(\mathfrak{C})$ the space of forms for which this norm is finite. Next let A be a bounded operator on $\mathfrak{C}$. We define the product $[F]A$ to be the form determined by

$$[F]A(f, g) = [F](Af, g). \tag{3.5}$$

Then clearly

$$\|[F]A\| < \|[F]\| \, \|A\|. \tag{3.6}$$

So far the Banach space $\mathfrak{G}$ was independent of our Hilbert space $\mathfrak{H}$. Now we impose our first requirement, namely, that both $\mathfrak{G}$ and $\mathfrak{H}$ can be embedded in a metric space $\mathfrak{M}$ in such a manner that

$$\mathfrak{G} \cap \mathfrak{H} \text{ is dense in } \mathfrak{H} \text{ and in } \mathfrak{G}. \tag{3.7}$$

Clearly an operator T in $\mathfrak{H}$ defines a form on $\mathfrak{D}(T) \cap \mathfrak{G} \times \mathfrak{D}(T) \cap \mathfrak{G}$; namely the form

$$[T]_{\mathfrak{G}}(f, g) = [T]_{\mathfrak{H}}(f, g) = (Tf, g). \tag{3.8}$$

In view of assumption (3.7) a sufficient condition for this domain to be dense in $\mathfrak{G}$ is that,

$$T \in \mathfrak{B}(\mathfrak{H}). \tag{3.9}$$

The closure of this form may or may not be in $\mathfrak{F}(\mathfrak{G})$. If it is, we denote it by the same symbol $[T]_{\mathfrak{G}}$. In this case we say that the operator T determines a form in $\mathfrak{F}(\mathfrak{G})$. If in addition to assumption (3.9)

$$T(\mathfrak{H} \cap \mathfrak{G}) \subset \mathfrak{G}, \tag{3.10}$$

and the closure of this operator is in $\mathfrak{B}(\mathfrak{G})$ we denote it by $T_{\mathfrak{G}}$. In this case we say that the operator T determines an operator in $\mathfrak{B}(\mathfrak{G})$.

These definitions allow us to state our key definition.

Definition 3.1. *The family of operators $A_0(\mu)$ is an approximating family to the given operator A over the given interval $\mathscr{I}$ if there are open regions $\mathscr{R}_\pm(\mathscr{I})$ of the form* $(3.1)_\pm$ *such that for each μ in $\mathscr{R}_\pm(\mathscr{I})$,*

$$\mu \in \rho(A_0(\mu)), \quad \text{i.e.,} \quad R(\mu, A_0(\mu) \in \mathfrak{B}(\mathfrak{H}). \tag{3.11}$$

Furthermore there is a space $\mathfrak{G}$ satisfying assumption (3.7) *such that with reference to it the two conditions that follow hold.*

Condition $G_1(\mathscr{I})$. *For each μ in $\mathscr{R}_\pm(\mathscr{I})$ the approximate resolvent operator, $R_\pm(\mu, A_0(\mu))$ determines a sesquilinear form in $\mathfrak{F}(\mathfrak{G})$ for which*

$$\sup_{\mu \in \mathscr{R}+(\mathscr{I})} \|[R(\mu, A_0(\mu))]_{\mathfrak{G}}\| < \infty. \tag{3.12}$$

Condition $G_2(\mathscr{I})$. *For each μ in $\mathscr{R}_\pm(\mathscr{I})$ the operator,*

$$T(\mu) = (A - A_0(\mu))R(\mu, A_0(\mu) \quad \text{in} \quad \mathfrak{H} \tag{3.13}$$

determines an operator in $\mathfrak{B}(\mathfrak{G})$; that is,

$$(T(\mu))_{\mathfrak{G}} \in \mathfrak{B}(\mathfrak{G}). \tag{3.14}$$

These operators depend norm-continuously on τ and admit continuous extensions on to the closures $\overline{\mathscr{R}_\pm(\mathscr{I})}$.

An example of Pavlov and Petras [10] concerning Hölder gentle perturbations implies that the existence of a family of approximating operators alone is not a sufficient condition for absolute continuity. Therefore, in analogy with such perturbations [5, 15], we introduce an additional condition. In it for each ω in $\mathscr{I}$ we set

$$T_\pm(\omega)_{\mathfrak{G}} = \lim_{\epsilon \to +0} T(\omega \pm i\epsilon)_{\mathfrak{G}} \tag{3.15}$$

where the right member is defined by relation (3.13) and according to Condition $\mathfrak{G}_2(\mathscr{I})$ this limit exists.

Condition $A(\mathscr{I})$. *For each ω in $\mathscr{I}$ each of the two limit operators $(I - T_\pm(\omega))_{\mathfrak{G}}$, admits an inverse in $\mathfrak{B}(\mathfrak{G})$.*

The theorem that follows formulates an absolute continuity criterion with the aid of these conditions.

Theorem 3.1. *Let A be a given self-adjoint operator and let $\mathscr{I}$ be a given compact interval. Suppose that over the interval $\mathscr{I}$, A admits a family of approximating operators in the sense of Definition* 3.1. *Suppose, further, that Condition $A(\mathscr{I})$ holds and that for each μ in $\mathscr{R}_\pm(\mathscr{I})$*

$$T(\mu) \text{ is compact in } \mathfrak{B}(\mathfrak{H}). \tag{3.16}$$

Then $A(\mathscr{I})$, the part of A over the interval $\mathscr{I}$, is absolutely continuous.

This theorem is a simplified version of a theorem formulated elsewhere [8b]. There, assumption (3.16) was replaced by Condition $A_2(\mathscr{I})$ which required that

$$[R(\mu, A)]_{\mathfrak{G}} = [R(\mu, A_0(\mu)]_{\mathfrak{G}}(I - T(\mu))_{\mathfrak{G}}^{-1}. \tag{3.17}$$

Accordingly we prove Theorem 3.1 be proving that assumption (3.16) implies relation (3.17). Clearly, assumption (3.16) and Condition $G_2(\mathscr{I})$ together show that

$$T(\mu)_{\mathfrak{G}} = T(\mu)_{\mathfrak{H}} \quad \text{on} \quad \mathfrak{H} \cap \mathfrak{G}. \tag{3.18}$$

Here and in the following the subscript $\mathfrak{H}$ emphasizes the fact that a given operator acts on $\mathfrak{H}$. Note that this subscript is in addition to the fact that $T(\mu)$ is defined in $\mathfrak{H}$. Since the spectrum of a self-adjoint operator is real [16], assumption (3.8) yields

$$\mu \in \rho(A) \cap \rho(A_0(\mu)).$$

It is not difficult to show that this fact together with assumption (3.16) yields

$$(I - T(\mu))^{-1} = (I - T(\mu))_{\mathfrak{H}}^{-1} \in \mathfrak{B}(\mathfrak{H}) \tag{3.19}$$

and

$$R(\mu, A)_{\mathfrak{H}} = R(\mu, A_0(\mu))_{\mathfrak{H}} \cdot (I - T(\mu))_{\mathfrak{H}}^{-1}. \tag{3.20}$$

Let $[R(\mu A_0(\mu))]_{\mathfrak{H}}$ denote the sesquilinear form of the operator $R(\mu, A_0(\mu))_{\mathfrak{H}}$. Then by definition

$$[R(\mu, A_0(\mu))]_{\mathfrak{H}} = [R(\mu, A_0(\mu))]_{\mathfrak{G}} \quad \text{on} \quad \mathfrak{H} \cap \mathfrak{G} \times \mathfrak{H} \cap \mathfrak{G}. \tag{3.21}$$

Relations (3.18), (3.19) and Condition $A(\mathscr{I})$ together yield

$$(I - T(\mu))_{\mathfrak{H}}^{-1} = (I - T(\mu))_{\mathfrak{G}}^{-1} \quad \text{on} \quad (I - T(\mu))_{\mathfrak{G}}(\mathfrak{H} \cap \mathfrak{G}). \tag{3.22}$$

Inserting relations (3.21) and (3.22) in relation (3.20) we obtain

$$\begin{aligned} [R(\mu, A)]_{\mathfrak{H}} = [R(\mu, A_0(\mu))]_{\mathfrak{G}}(I - T(\mu))_{\mathfrak{G}}^{-1}, \quad &\text{on} \\ \text{on} \quad (I - T(\mu))_{\mathfrak{G}}(\mathfrak{H} \cap \mathfrak{G}) \times (I - T(\mu))_{\mathfrak{G}}(\mathfrak{H} \cap \mathfrak{G}). \end{aligned} \tag{3.23}$$

Since a bounded operator with a bounded inverse maps an arbitrary dense set onto a dense set, Condition $A(\mathscr{I})$ implies

$$(I - T(\mu)_{\mathfrak{G}}(\mathfrak{H} \cap \mathfrak{G}) \quad \text{is dense in} \quad \mathfrak{G}. \tag{3.24}$$

This fact, Condition $G_1(\mathscr{I})$, and another application of Condition $A(\mathscr{I})$ allows us to extend relation (3.23) by closure to all of $\mathfrak{G} \times \mathfrak{G}$. This way we arrive at the validity of relation (3.17). This completes the proof of Theorem 3.1.

4. A lemma on approximating potentials

Let $p(\mu)$ be a given family of potentials and let $\mathscr{I}$ be a given interval. Recall that definitions (2.8) and (2.9) assign to each $p(\mu)$ an operator $L(p(\mu))$. In this section we formulate conditions which ensure that this family of operators approximates the operator $L(p)$ over the interval $\mathscr{I}$. These conditions are simplified versions of conditions formulated elsewhere [9]. These simplifications are partly due to the simplifiying assumption (2.2) on the short range potential p_1 and partly due to the fact that assumptions (2.10) and (2.11) imply that

$$\lim_{\xi \to \infty} p_2(\xi) = 0.$$

Condition $I(\mathscr{I})$. *There are regions $\mathscr{R}_\pm(\mathscr{I})$ of the form* (3.1)$_\pm$ *such that for each μ in these regions the operator $L(p(\mu))$ satisfies assumption* (3.11). *This family of potentials is related to the original potential p by the estimate*

$$\int_0^\infty \sup_{\mathscr{R}_+(\mathscr{I})} |(p - p(\mu))(\xi)| \, d\xi < \infty. \tag{4.1}$$

Furthermore, for each point ω of $\mathscr{I}$ each of the two limit functions exists

$$\lim_{\epsilon \to +0} p(\omega \pm i_\epsilon)(\xi) = p_\pm(\omega)(\xi), \tag{4.2}$$

and this convergence is uniform in ω in $\mathscr{I}$ and ξ in any compact subset of $\mathscr{R}^+$.

Condition II$(\mathscr{I})$. *The family of approximate resolvents, $R(\mu, L(p(\mu)))$, is such that their kernels satisfy the estimate*

$$\sup_{\mu \in \mathscr{R}_+(\mathscr{I})} \sup_{(\xi, \eta) \in \mathscr{R}^+ \times \mathscr{R}^+} |R(\mu, L(p(\mu)))(\xi, \eta)| < \infty. \tag{4.3}$$

Furthermore, for each point ω of $\mathscr{I}$ each of the two limit kernels exists

$$\lim_{\varepsilon \to +0} R(\omega + i\varepsilon, L(p(\omega + i\varepsilon)))(\xi, \eta) = R_\pm(\omega, L(p((\omega)))(\xi, \eta), \tag{4.4}$$

and this convergence is uniform in ω in $\mathscr{I}$ and (ξ, η) in any compact subset of $\mathscr{R}^+ \times \mathscr{R}^+$.

In the following lemma we use these conditions to formulate conditions ensuring that the family of operators $L(p(\mu))$ approximates the operator $L(p)$. Recall that in Definition 3.1 this approximation property was stated with reference to a given interval $\mathscr{I}$ and with reference to a given space $\mathfrak{G}$. In this following lemma we define such a space by defining a norm on the space of measurable functions.

Lemma 4.1. *Let p be a given potential and let $p(\mu)$ be a given family of potentials which satisfies Condition $I(\mathscr{I})$ and $II(\mathscr{I})$. Suppose that the operator*

$$T(\mu) = (L(p) - L(p(\mu)))R(\mu, L(p(\mu))) \tag{4.5}$$

satisfies assumption (3.9) *with* $\mathfrak{H} = \mathfrak{L}_2(\mathscr{R}^+)$. *Define a function n by*,

$$n(\xi) = \sup_{\mu \in \mathscr{R}_+(\mathscr{I})} |(p - p(\mu))(\xi)| + \exp(-\xi) \tag{4.6}$$

and define a norm $\mathfrak{G}$ *by*

$$\|f\|_{\mathfrak{G}} = \left\| \left(\frac{1}{n}\right)^{1/2} f \right\|_{\mathfrak{H}}.$$

Suppose further that the operator $T(\mu)$ *satisfies assumption* (3.10) *with reference to this space* $\mathfrak{G}$. *Then the family of operators* $L(p(\mu))$ *approximates the operator* $L(p)$ *over the interval* $\mathscr{I}$ *with reference to this norm.*

The two remarks that follow were used in the proof of the original version of Lemma 4.1 [9] and will be used in Section 6.

First we let the transformation $M(1/n)^{1/2}$ mapping $\mathfrak{G}$ into $\mathfrak{H}$ be defined by,

$$M\left(\frac{1}{n}\right)^{1/2} f(\xi) = \left(\frac{1}{n(\xi)}\right)^{1/2} f(\xi), \quad f \in \mathfrak{G}. \tag{4.8}$$

Then we remark that according to definition (4.7) this is an isometry mapping $\mathfrak{G}$ onto all of $\mathfrak{H}$. Hence it is a unitary transformation and clearly the inverse is given by,

$$M(n^{1/2})f(\xi) = n^{1/2}(\xi)f(\xi), \quad f \in \mathfrak{H}. \tag{4.9}$$

Second let $M(n^{1/2})$ denote the operator in $\mathfrak{H}$ with domain,

$$\mathfrak{D}(M(n^{1/2})) = \{f : f \in \mathfrak{H}, n^{1/2} f \in \mathfrak{H}\}. \tag{4.10}$$

Suppose that T is a given operator such that,

$$T \in \mathfrak{B}(\mathfrak{H}) \quad \text{and} \quad M(n^{1/2})TM\left(\frac{1}{n}\right)^{1/2} \in \mathfrak{B}(\mathfrak{H}), \tag{4.11}$$

where we denote an operator and its closure by the same symbol. Then we remark that relation (4.11) implies that $T_{\mathfrak{G}}$ is in $\mathfrak{B}(\mathfrak{G})$ and that this operator is unitarily equivalent to the second term in (4.11); that is,

$$T_{\mathfrak{G}} \sim M(n^{1/2})TM\left(\frac{1}{n}\right)^{1/2}.$$

5. Construction of a family of Jeffreys-Wentzel-Kramers-Brillouin—approximate potentials

Let $\mathscr{I}$ be a given compact subinterval of $\mathscr{R}^+$ and let p be the potential of Theorem 2.1. In this section with the aid of the JWKB approximation method [12, 13, 18] we construct a family of approximate potentials $q(\mu)$. That is to say, this family of potentials is such that over the interval $\mathscr{I}$ the family of operators $L(q(\mu))$ approximates the operator $L(p)$ in the sense of Definition 3.1. Note that in this section we give the construction only and postpone the proof of the approximation property until the next section.

To construct such a family of approximate potentials $q(\mu)$ we need a family of approximate solutions to the family of equations,

$$(\mu - L(p))f(\mu) = 0. \tag{5.1}$$

More specifically let $y(\mu)$ be an approximate solution to this equation. Then we define a corresponding approximate potential $q(\mu)$ by the requirement that

$$(\mu - L(q(\mu)))y(\mu) = 0. \tag{5.2}$$

The technical Definition 3.1 has been motivated by scattering theory. Since scattering takes place far away from the scattering center we are interested in approximating the solutions of equation (5.1) for large values of the independent variable. Inserting definition (2.9) in equation (5.1) we obtain the more detailed equation

$$\hbar^2 f''(\mu) + (\mu + p)f(\mu) = 0. \tag{5.3}$$

For a moment let us assume that the potential is a constant and set

$$\mu - p = \alpha. \tag{5.4}$$

Then equation (5.2) yields

$$\hbar^2 f'' + \alpha f = 0. \tag{5.5}$$

Next we replace Plank's constant $\hbar$ by a small parameter h in this equation,

$$h^2 g_h'' + \alpha g_h = 0. \tag*{(5.5)h}$$

Clearly one can obtain a solution of (5.5) from a solution of $(5.5)_h$ by a simple scaling of the independent variable. Specifically by setting

$$f(\xi) = g_h\left(\frac{h}{\hbar}\xi\right). \tag{5.6}$$

This formula shows that the value of a solution to (5.5) at some large value of ξ equals the value of a solution to $(5.5)_h$ at some fixed value of ξh and hence small value of h. Now it is perfectly possible that for a suitable class of potentials instead of a strict equality in relation (5.6) we have an asymptotic equality. At present we leave open the question of a suitable class.

We analyze, in a formal manner, for small values of h, the solutions of the equation,

$$h^2 g_h''(\mu) + (\mu - p)g_h = 0. \tag*{$(5.7)_h$}$$

As usual in the theory of the JWKB-approximation method [11, 12, 18], we seek the solution in the form

$$g_h(\mu)(\xi) = \exp\int_0^\xi w_h(\mu)(\sigma)\,d\sigma. \tag*{$(5.8)_h$}$$

Then elementary algebra shows that

$$g_h''(\mu) = (w_h'(\mu) + w_h^2(\mu))g_h(\mu). \tag*{$(5.9)_h$}$$

Inserting this relation in equation $(5.7)_h$ we obtain the Ricatti equation

$$(w_h'(\mu) + w_h^2(\mu))h^2 + (\mu - p) = 0. \tag*{$(5.10)_h$}$$

The lemma that follows is a version of a result of Sibuya [4, 19] and describes the coefficients of h in the formal power series expansion of $w_h(\mu)$. In it we need a notation for the space of polynomials of mixed homogeneity. Specifically for each $n = 0, 1, \ldots,$ we set,

$$\mathfrak{P}_{n+1}(k_0 + 2k_1 + \cdots + (n+1)\,k_n = n+1) = \{p{:}p(s_0, \ldots, s_n) \\ = \sum \alpha(k_0, k_1, \ldots, k_n)\, s_0^{k_0}\, s_1^{k_1} \cdots s_n^{k_n}\}, \tag{5.11$_{n+1}$}$$

where the summation on the right is extended over those positive integers or zero which satisfy the equation on the left.

Lemma 5.1. *For each of the two branches of* $(p - \mu)^{1/2}$, *equation* $(5.9)_h$ *admits a formal solution of the form,*

$$w_h(\mu) = (p - \mu)^{1/2}\frac{1}{h} + a_0(\mu) + \sum_{n=1}^{\infty} a_n(\mu)h^n. \tag{5.12$_h$}$$

Furthermore,

$$a_0(\mu) = -\frac{1}{4}\left(\frac{p'}{p - \mu}\right) \tag{5.13$_0$}$$

and for each $n = 1, 2, \ldots,$ *the coefficient* $a_n(\mu)$ *has the property that there is a polynomial* $\bar{p}_n$ *such that*

$$a_n(\mu) = \left(\frac{1}{p - \mu}\right)^{n/2} \bar{p}_n(a_0(\mu), a_0'(\mu), \ldots, a_0^{(n)}(\mu)) \tag{5.13$_n$}$$

and

$$\bar{p}_n \in \mathfrak{P}_{n+1}(k_0 + 2k_1 + \cdots + (n+1)\,k_n = n+1). \tag{5.14$_n$}$$

We start the proof of this lemma by setting,

$$q_h(\mu) = (w_h'(\mu) + w_h^2(\mu))\,h^2 + \mu. \tag{5.15$_h$}$$

Then using this notation equation (5.10), can be written as,

$$p - q_h(\mu) = 0. \tag{5.16$_h$}$$

To prove conclusion $(5.12)_h$ we insert its right member in definition $(5.15)_h$ and write the result as a formal power series in h. Inserting this formal power series, in turn, in equation $(5.16)_h$ we see that the lowest power of h equals one. The requirement that the coefficient of h be zero, yields the following equation for $a_0(\mu)$,

$$2(p - \mu)^{1/2}\, a_0(\mu) - \frac{1}{2}\frac{p'}{(p - \mu)^{1/2}} = 0. \tag{5.17$_0$}$$

The requirement that the coefficient of h^2 be zero yields the following equation for $a_1(\mu)$,

$$2(p - \mu)^{1/2}\, a_1(\mu) + a_0^2(\mu) + a_0'(\mu) = 0. \tag{5.17$_1$}$$

In fact, in general, for $n = 1, 2, \ldots,$

$$2(p - \mu)^{1/2}\, a_n(\mu) + \sum_{j+k=n-1} a_j(\mu)a_k(\mu) + a_{n-1}'(\mu) = 0, \tag{5.17$_n$}$$

where in the summation j and k take on independently positive integer values and possibly zero, the only requirement being that their sum is $n - 1$. Clearly these equations do admit solutions and the formal power series corresponding to them are formal solutions of equation $(5.10)_h$. This completes the proof of conclusion $(5.12)_h$.

To prove conclusion $(5.13)_0$ we solve equation $(5.17)_0$ for the unknown function $a_0(\mu)$. To prove conclusion $(5.13)_n$ for general $n = 1, 2, \ldots$, we solve equation $(5.17)_n$ for the unknown function $a_n(\mu)$. This yields,

$$a_n(\mu) = \frac{1}{(p - \tau)^{1/2}} \frac{1}{2}\left[\left(\sum_{j+k=n-1} a_h(\mu)a_k(\tau)\right) + a'_{n-1}(u)\right], \tag{$(5.18)_n$}$$

where in the summation j and k range over the same set of values as in relation $(5.17)_n$.

To complete the proof of conclusion $(5.13)_1$ we define the polynomial $\bar{p}_1$ by

$$\bar{p}_1(s_0, s_1) = -\frac{1}{2}(s_0^2 + s_1). \tag{$(5.19)_1$}$$

Then inserting this definition in formula $(5.18)_1$ we see the validity of this conclusion.

To complete the proof of conclusion $(5.13)_2$ we differentiate the already established conclusion $(5.13)_1$. After an elementary algebra and the use of conclusion $(5.13)_0$ this yields

$$a'_1(\mu) = \frac{1}{(p - \mu)^{1/2}} [2a_0(\mu)\bar{p}_1(a_0(\mu), a_0^1(\mu)) + \bar{p}_1(a_0(\mu), a'_0(\mu))],$$

where

$$\bar{p}_1(a_0(\mu), a'_0(\mu)) = \partial_0\bar{p}_1(a_0(\mu), a'_0(\mu))\cdot a'_0(\mu) + \partial_1\bar{p}_1(a_0(\mu), a'_0(\mu))a''_0(\mu),$$

and ∂_0, ∂_1, denote the partial derivatives of $\bar{p}_1$, with reference to the zeroth and first arguments. Inserting these formulas in formula $(5.18)_2$ we obtain

$$a_2(\mu) = \frac{1}{(p - \mu)} [-2a_0(\mu)\bar{p}_1(a_0(\mu), a'_0(\mu)) - \frac{1}{2}\bar{p}'_1(a_0)\mu), a'_0(\mu))]. \tag{5.20}$$

Hence, defining the polynomial $\bar{p}_2$ by

$$\bar{p}^2(s_0, s_1, s_2) = -2s_0\bar{p}_1(s_0, s_1) - \frac{1}{2}(\partial_0\bar{p}_1(s_0, s_1)s_1 + \partial_1\bar{p}_1(s_0, s_1)s_2) \tag{$(5.19)_2$}$$

we obtain the validity of conclusion $(5.13)_2$.

To complete the proof of conclusion $(5.13)_n$ for an arbitrary positive integer n we proceed by induction. In fact, we define the polynomial $\bar{p}_n$ by

$$\begin{aligned}\bar{p}_n(s_0, \ldots, s_n) = {} & -ns_0\bar{p}_{n-1}(s_0, \ldots, s_{n-1}) - \frac{1}{2}\left(\sum_{j=0}^{n-1} \partial_j\bar{p}_{n-1}(s_0, \ldots, s_{n-1})s_{j+1}\right) \\ & -\frac{1}{2}\sum_{j+k=n-1} \bar{p}_j(s_0, \ldots, s_j)\cdot\bar{p}_k(s_0, \ldots, s_k).\end{aligned} \tag{$(5.19)_n$}$$

Note that in the second summation, in contrast to formula $(5.17)_n$, j and k range over strictly positive values. This is in accordance with the fact that the polynomials

$\bar{p}_j$ have been defined for $j = 1, 2, \ldots, n-1$. For brevity we do not carry out the details of this induction and consider the proof of conclusion $(5.13)_n$ complete.

The validity of conclusion $(5.14)_1$ is immediate from definitions $(5.11)_2$ and $(5.19)_1$. Inserting conclusion $(5.14)_1$ in definition $(5.19)_2$ we see the validity of conclusion $(5.14)_2$. Similarly we see the validity of conclusion $(5.14)_n$ from the inductive definition $(5.19)_n$. For brevity we do not prove this fact and consider the proof of Lemma 5.1 complete.

Having established Lemma 5.1 we return to the question of constructing a family of approximate potentials. We shall do this with the aid of the formal power series of Lemma 5.1. Motivated by assumption (2.10) we truncate the series $(5.12)_h$ after three terms and replace the small parameter h by Planck's constant $\hbar$. That is to say, we set

$$w(\mu) = (p - \mu)^{1/2}\frac{1}{\hbar} + a_0(\mu) + a_1(\mu)\hbar. \tag{5.21}$$

Next we choose a branch of the square root function by the requirement that

$$\operatorname{Re}\sqrt{z} \geq 0 \quad \text{for} \quad z \notin (-\infty, 0]. \tag{5.22}$$

Then we choose two single valued branches, $\pm\sqrt{p-\mu}$, of the double valued function $(p-\mu)^{1/2}$. Inserting these branches in definition (5.21) yields the two functions,

$$w^{\pm}(\mu) = \pm\sqrt{p-\mu}\,\frac{1}{\hbar} + a_0(\mu) \pm a_1(\mu)\hbar. \tag{5.23$^{\pm}$}$$

Finally setting

$$w_h(\mu) = w^-(\mu) \quad \text{and} \quad h = \hbar$$

in definition $(5.15)_h$ we obtain the family of approximate potentials,

$$q(\mu) = (w^-(\mu)' + w^-(\mu)^2)\hbar^2 + \mu. \tag{5.24}$$

Remembering definition (2.9) and the fact that relation $(5.8)_h$ implies relation $(5.9)_h$ we see that the function,

$$y(\mu)(\xi) = \exp\left(\int_0^{\xi} w^-(\mu)(\sigma)\,d\sigma\right), \tag{5.25}$$

satisfies equation (5.2) where $q(\mu)$ is given by (5.24). In fact, this property motivated our definition (5.24).

Note that in this section the role of the functions $w^{\pm}(\mu)$ is symmetric. We have chosen the function $w^-(\mu)$ versus $w^+(\mu)$ since this choice is more convenient for the estimates of Section 6. Also note that in contrast to the case of the second order approximation,

$$w^-(\mu)' + w^-(\mu)^2 \neq w^+(\mu)' + w^+(\mu)^2.$$

Hence replacing $w^-(\mu)$ by $w^+(\mu)$ in definition (5.25) the resulting function will not satisfy equation (5.2) with the $q(\mu)$ of definition (5.24).

Recall that Definition 3.1 is of a rather technical nature. This suggests that in order that the family of potentials of definition (5.24) approximate the potential p of Theorem 2.1, rather technical assumptions on p will be required. Actually all that we need is that these potentials approximate a long range part of this potential.

Accordingly from now on we set

$$p = p_2,$$

in the previous formulae. This, at least, allows us to carry out the indicated differentiation in definition (5.24). As to be expected these assumptions will have to ensure that for Reμ in $\mathscr{I}$ the JWKB-approximate solution of definition (5.25) has no turning points. That is to say, there is no point ξ in $\mathscr{R}^+$ such that

$$p_2(\xi) - \mathrm{Re}\mu = 0.$$

After these preparations we formulate these assumptions.

Condition 0($\mathscr{I}$). *The potential p_2 satisfies the assumptions of Theorem* 2.1 *and in addition it is such that*

$$\mathrm{dist}\,(\mathscr{I}, p^2(\mathscr{R}^+)) \neq 0. \tag{5.26}$$

Furthermore, the family of approximate potentials $q(\mu)$ is such that

$$\int \sup_{\mu\in\mathscr{R}_+(\mathscr{I})} |(p_2 - q(\mu))|(\xi)\, d\xi < \infty, \tag{5.27}$$

and the family of functions $w^+(\mu)$ of definition (5.23)$^+$ *is such that*

$$\inf_{\mu\in\mathscr{R}_+(\mathscr{I})} \inf_{\xi\in\mathscr{R}^+} \mathrm{Re}((w^+(\mu) - a_0(\mu)(\xi))) \geq 0. \tag{5.28}$$

Note that Condition 0($\mathscr{I}$) is a condition on p_2 inasmuch as the left member of each of the assumptions (5.26), (5.27), (5.28), is defined in terms of p_2. We do not claim and it is not true that if the potential p satisfies the assumptions of Theorem 2.1 then for any decomposition its long range part satisfies Condition 0($\mathscr{I}$). All that we claim is that such a potential admits a decomposition with this property. This is described in more specific terms in the lemma that follows.

Lemma 5.2. *Let $\mathscr{I}$ be a given compact interval which does not contain zero. Suppose that the potential p satisfies the assumptions of Theorem* 2.1. *Then p admits a decomposition of the form* (2.7) *such that p_1 is short range and p_2 satisfies* Condition 0($\mathscr{I}$).

To construct a potential p_2 satisfying assumption (5.26) let p be any given long range potential. Then from the assumptions on $\mathscr{I}$ and from the fact that for a long range potential $p(\infty) = 0$, we see that infinity has a neighborhood, $(\tilde{\xi}, \infty)$, such that

$$\mathrm{dist}\,(\mathscr{I}, p((\tilde{\xi}, \infty))) \neq 0. \tag{5.29}$$

It is an elementary fact that from the interval $(\tilde{\xi} + 1, \infty)$ this potential can be extended to all of $\mathscr{R}^+$ in such a manner that the extended potential is also smooth and denoting it by p_2,

$$p_2(\mathscr{R}^+) \subset p((\tilde{\xi} + 1, \infty)).$$

Inserting this relation in (5.29) we obtain that the potential p_2 satisfies assumption (5.26).

Next we show that this potential p_2 also satisfies assumption (5.27). For this purpose recall definition (5.24) and assumption (2.10). Together with the already established assumption (5.26) they show that assumption (5.27) is implied by the estimate,

$$(p_2 - q(\mu))(\xi) = 0\left(\frac{1}{\xi}\right)^{\beta + 3(1-\gamma)} \quad at \quad \xi = \infty, \tag{5.30}$$

uniformly in μ in $\mathscr{R}_{\pm}(\mathscr{I})$. We start the proof of this estimate by recalling definitions (5.21), (5.24) and the way we arrived at equations $(5.17)_0$ and $(5.17)_1$. Combining them we see that,

$$p_2 - q(\mu) = -(2a_0(\mu)a_1(\mu) + a_1'(\mu))\hbar^3 - a_1(\mu)^2\hbar^4. \tag{5.31}$$

Equation $(5.17)_2$ and conclusion $(5.13)_2$ of Lemma 5.1 together show that

$$-(2a_0(\mu)a_1(\mu) + a_1'(\mu)) = \frac{2}{(p_2 - \mu)^{1/2}}\bar{p}_2(a_1(\mu), a_0'(\mu), a_0''(\mu)).$$

Inserting this relation and conclusion $(5.13)_1$ of Lemma 5.1 in relation (5.31) we obtain,

$$p_2 - q(\mu) = \frac{2\hbar^3}{(p_2 - \mu)^{1/2}}\bar{p}_2(a_0(\mu), a_0'(\mu), a_0''(\mu)) - \frac{\hbar^4}{(p_2 - \mu)}\bar{p}_1(a_0(\mu), a_0'(\mu))^2. \tag{5.32}$$

We complete the proof of estimate (5.30) by estimating each of the two terms on the right. For this purpose we introduce three notations. First we denote by $\mathfrak{A}\,(p_2 - \mu)^{-1/2})$ the algebra generated by the function $(p_2 - \mu)^{-1/2}$. In other words,

$$\mathfrak{A}((p_2 - \mu)^{-1/2}) = \text{Span}\,\{(p_2 - \mu)^{-1/2}, (p_2 - \mu)^{-1}, (p_2 - \mu)^{-3/2}, \ldots\}, \tag{5.33}$$

where the right-member consists of all finite linear combinations of the functions in the bracket. Secondly we refer to each of the two functions $(p_2')^2$ and p_2'' as second order in p_2 and for brevity we set,

$$0^2(p_2) = \{(p_2')^2, p_2''\}. \tag{5.34}$$

In general we define the n-th order terms in p_2 to be the set of functions,

$$0^n(p_2) = \{(p_2')k_1(p_2'')^k2(p_2''')^k3 \ldots (p_2{}^{(n)})^kn\}, \tag{5.34^n}$$

where these indices are restricted by the requirement that,

$$k_1 + k_2 + k_3 + \cdots + k_n = n.$$

Thirdly, with the aid of these two notations we set,

$$\mathfrak{L}(0^n(p_2), \mathfrak{A}((p_2 - \mu)^{-1/2}) = \text{Span}\,\{0^n(p_2)\}, \tag{5.35^n}$$

where the right member consists of all finite linear combinations of elements of $0^n(p_2)$ with coefficients from the algebra $\mathfrak{A}(p_2 - \mu)^{-1/2})$. Combining definition $(5.34)^n$ with conclusion $(5.13)_0$ of Lemma 5.1 we see that for each positive integer n,

$$a_0^{(n)}(\mu) \in 0^{n+1}(p_2). \tag{5.36^n}$$

Combining this relation, in turn, with conclusion $(5.13)_h$ of Lemma 5.1 and definition $(5.35)^{n+1}$, we see that

$$\bar{p}_n(a_0(\mu), a_0'(\mu), \ldots, a_0^{(n)}(\mu)) \in \mathfrak{L}(0^{n+1}(p_2), \mathfrak{A}((p_2 - \mu)^{-1/2}). \tag{5.37$_n$}$$

In particular, we see that,

$$\bar{p}_2(a_0(\mu), a_0'(\mu), a_0''(\mu)) \in \mathfrak{L}(0^3(p_2), \mathfrak{A}((p_2 - \mu)^{-1/2}). \tag{5.37$_2$}$$

Similarly, it follows that,

$$\bar{p}_1(a_0(\mu), a_0'(\mu))^2 \in \mathfrak{L}(0^4(p_2), \mathfrak{A}((p_2 - \mu)^{-1/2}). \tag{5.38}$$

Note that aside from a factor in the algebra $\mathfrak{A}((p - \mu)^{-1/2})$, these two polynomials are the coefficients of $\hbar^3$ and $\hbar^4$ in relation (5.32). Hence relations $(5.37)_2$ and (5.38) say that in relation (5.32) the coefficients of the third and fourth order terms in $\hbar$ are third and fourth order terms in p_2.

It is not difficult to show that assumptions (2.7) and (2.6) imply that for each positive integer n each of the n-th order terms in p_2 decays at infinity with exponent at least $\beta + n(1 - \gamma)$. Symbolically we express this fact as,

$$0^n(p_2)(\xi) = 0\left(\frac{1}{1 + \xi}\right)^{\beta + n(1-\gamma)} \quad \text{at} \quad \xi = \infty. \tag{5.39^n}$$

Inserting estimates $(5.39)^3$, $(5.39)^4$ and relations $(5.37)_2$, (5.38) in relation (5.32) we arrive at the validity of estimate (5.30). From this, in turn, we arrive at the validity of assumption (5.27) for the potential p_2.

Finally we show that this potential p_2 also satisfies assumption (5.28). For this purpose recall defintions $(5.19)_1$ and conclusion $(5.13)_0$ of Lemma 5.1. Combining them with an elementary algebra they show that,

$$\bar{p}_1(a_0(\mu), a_0'(\mu)) = \hbar\left(\frac{3}{16}\frac{p_2''}{(p_2 - \mu)} - \frac{1}{2}\frac{(p_2')^2}{(p_2 - \mu)^2}\right). \tag{5.40}$$

Combining relation (5.40), in turn, with definition $(5.23)^+$ and with conclusion $(5.13)_1$ of Lemma 5.1 shows that setting,

$$b(\mu) = \frac{1}{\hbar} + \hbar\left(\frac{3}{16}\frac{(p_2')^2}{(p_2 - \mu)^3} - \frac{1}{2}\frac{p_2''}{(p_2 - \mu)^2}\right), \tag{5.41}$$

we have,

$$w^+(\mu) - a_0(\mu) = \sqrt{p_2 - \mu}\, b(\mu). \tag{5.42}$$

It is an elementary fact that for each compact interval $\mathscr{I}$ which does not contain zero and for each positive integer n,

$$\sup_{\mathrm{Re}\, z \in (\mathscr{I})} \left|\frac{\mathrm{Re}\,((1/z)^n \sqrt{z})}{\mathrm{Re}\,(\sqrt{z})}\right| < \infty. \tag{5.43^n}$$

The already established assumption (5.26) allows us to apply estimates $(5.43)^2$ and $(5.43)^3$ to the complex number

$$z = p_2(\xi) - \mu.$$

Remembering estimate $(5.39)^2$ this yields the existence of a number ξ_0 such that

$$\inf_{\mu \in \mathscr{R}_+(\mathscr{I})} \inf_{\xi \in [\xi_0, \infty]} \mathrm{Re}\,(w^+(\mu) - a_0(\mu))(\xi) \geq 0.$$

In other words assumption (5.28) holds in a neighborhood of infinity. The fact that it holds over all of $\mathscr{R}^+$ follows by possibly changing the definition of p_2 over the interval $[0, \xi_0]$. For brevity we omit the details of this construction and consider the proof of Lemma 5.2 complete.

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