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The Relativistic Two Body Problem

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Abstract. The two body problem of relativistic mechanics with scalar interaction is treated in the framework of a canonical formalism which is manifestly covariant.

To each solution of the reduced Newtonian two body problem with radial potential, with the one restriction that the speeds stay smaller than the velocity of light, there corresponds in our model a solution differing from the Newtonian one only by a global Lorentz transformation. Other solutions occur which show properly relativistic effects in particular the potential energy can become bigger than the masses.

Introduction

The difficulties encountered in relativity to elaborate a canonical dynamics which is covariant and non-trivial are well known. Let us simply remember the no go theorems of D. G. Currie [1]. Under such conditions it is only possible to overcome these difficulties by accepting to change radically the point of view of Einstein's theory. In fact the relativistic dynamics that we proposed before [2] avoids these difficulties.

In this paper we study in detail the relativistic two body problem corresponding to Newton's classical case. We hope to prove by this example the soundness of our theory and its capacity to describe the real phenomena of relativistic physics.

Before we enter into our model's details we would like to recall to mind the essential differences between our point of view and the usual point of view. In the usual Einstein theory each particle is identified with a trajectory in space-time and the dynamics of the system is simply reduced to a description of these trajectories. For instance in the free particle case, the system is reduced to a family of time-like straight lines. As nothing changes, since nothing runs over these trajectories, we cannot properly speak of system's behaviour, and the concept of probability, one of the characteristics of quantum theory, is meaningless in such a scheme.

This is the reason why, when taking seriously both Einstein's ideas and Newton's ideas, we have admitted the existence of a space-time supplied with the geometry given by Poincaré group and have identified each particle with a single point of that space-time, or an event according to Einstein. Moreover we have postulated the existence of another time, the historical time, which passes by uniformly and inexorably, as Newton imagined, and that we can neither change nor directly observe.

We are then led to describe the state of each particle by eight independent numbers

$$q = (q^1, q^2, q^3, q^4) = (\vec{q}, t)$$

$$p = (p^1, p^2, p^3, p^4) = (\vec{p}, E/c^2)$$

where c is the velocity of light.

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q is identified with the position in space-time and p with the state of motion, i.e. the energy momentum.

Under these conditions

$$p^2 = p_\mu p^\mu$$

is a function of the state whose value is susceptible to change during the system's behaviour.

The behaviour itself is described as a function of the historical time τ and is governed by the canonical equations

$$\frac{dq_i^\mu}{d\tau} = g^{\mu\nu} \frac{\partial K}{\partial p_i^\nu}$$

and

$$\frac{dp_i^\mu}{d\tau} = -g^{\mu\nu} \frac{\partial K}{\partial q_i^\nu}$$

where $K = K(p_i^\mu, q_i^\mu)$ is a scalar function of the state $(p_{i'}^\mu, q_i^\mu)$ of each particle i.

In the following we note

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu$$

for the four-dimensional scalar product and we have chosen, as in [2], for the metric $g_{\mu\nu} = (1, 1, 1, -c^2)$.

The Two Body Problem

In the case of the two body problem, our model consists of taking:

$$K(p_1^\mu, q_1^\mu, p_2^\mu, q_2^\mu) = \frac{p_1^2}{2M_1} + \frac{p_2^2}{2M_2} + \Phi(|q_1 - q_2|) \tag{1}$$

where $|q_1 - q_2| = \sqrt{(q_1 - q_2)^2}$, M_1 and M_2 are constants which are the masses of the particles 1 and 2 and Φ a function which characterizes the interaction between both particles.

In such a case it is convenient to use the following new variables

$$\begin{aligned} P &= p_1 + p_2, & Q &= \frac{M_1 q_1 + M_2 q_2}{M_1 + M_2} \\ p &= \frac{M_1 p_2 - M_2 p_1}{M_1 + M_2}, & q &= q_2 - q_1 \end{aligned} \tag{2}$$

They define a canonical transformation leaving K invariant.

It leads to the decomposition

$$K = K_0 + k$$

where

$$\begin{aligned} K_0 &= \frac{P^2}{2M}, & M &= M_1 + M_2 \\ k &= \frac{p^2}{2\mu} + \Phi(|q|), & \mu &= \frac{M_1 M_2}{M_1 + M_2} \end{aligned}$$

The motion is then governed by the equations

$$\frac{dQ^\mu}{d\tau} = \frac{P^\mu}{M}, \qquad \frac{dP^\mu}{d\tau} = 0$$
$$\frac{dq^\mu}{d\tau} = \frac{p^\mu}{\mu}, \qquad \frac{dp^\mu}{d\tau} = -g^{\mu\nu}\frac{\partial\Phi}{\partial q^\nu} \tag{3}$$

Therefore K, k and also $P = (\vec{P}, E/c^2)$ are first integrals. $\vec{P}$ is the total momentum of the system and E its total energy. The four vector $Q = (\vec{Q}, T)$ defines a point of space time called the center of mass. Then

$$T = \frac{E}{Mc^2}\tau$$

passes by uniformly and so Møller's condition on the time used by J. L. Cook [3] is justified by our model.

It is important to note from (3) that the antisymmetric tensor

$$j^{\mu\nu} = q^\mu p^\nu - p^\mu q^\nu$$

is also first integral. It is the generalized angular momentum of the system relative to the center of mass.

We write

$$\vec{a} = (j_{14}, j_{24}, j_{34}) = c^2 t\vec{p} - e\vec{q}$$
$$\breve{b} = (j_{23}, j_{31}, j_{12}) = \vec{q} \wedge \vec{p} \tag{4}$$

where $q = (\vec{q}, t)$ and $p = (\vec{p}, e/c^2)$.

Consequently $\varepsilon^{\mu\nu\rho\lambda}$ being the canonical antisymmetric tensor we have

$$\vec{a}\breve{b} = \tfrac{1}{4}\varepsilon^{\mu\nu\rho\lambda} j_{\mu\nu} j_{\rho\lambda} = 0$$
$$\breve{b}^2 - \vec{a}^2/c^2 = \tfrac{1}{2} j^{\mu\nu} j_{\mu\nu} \tag{5}$$

It can also be verified that the relative motion takes place in a plane $\mathscr{P}$ of the relative q^μ-coordinate space.

Moreover this plane is space-like (i.e. $q^2 > 0$, $\forall \in \mathscr{P}$) if and only if

$$j^{\mu\nu} j_{\mu\nu} > 0$$

that is $|\breve{b}| > |\vec{a}|/c$.

In the following we shall discuss this case. The other case where $j^{\mu\nu} j_{\mu\nu} \leqslant 0$ will be discussed afterwards.

So when $j^{\mu\nu} j_{\mu\nu} > 0$ there exists reference frames relative to the point defined by the center of mass in which $\vec{a} = 0$. Effectively by a pure Lorentz transformation of velocity $\vec{v}$ we have

$$\vec{a} \mapsto \vec{a}_{||} + \gamma(\vec{a}_\perp + \vec{v} \wedge \breve{b})$$
$$\breve{b} \mapsto \breve{b}_{||} + \gamma\left(\breve{b}_\perp - \frac{\vec{v} \wedge \vec{a}}{c^2}\right)$$

where $\vec{a}_{||}$, $\breve{\vec{b}}_{||}$ and $\vec{a}_{\perp}$, $\breve{\vec{b}}_{\perp}$ are respectively the parallel and perpendicular components of $\vec{a}$ and $\breve{\vec{b}}$ relatively to $\vec{v}$ and $\gamma = (1 - \vec{v}^2/c^2)^{-1/2}$.

Consequently those reference frames are characterized by the pure Lorentz transformations given by:

$$\vec{v} = \frac{\vec{a} \wedge \breve{\vec{b}}}{b^2} + \lambda \frac{\breve{\vec{b}}}{|\breve{\vec{b}}|}$$

λ such that $\vec{v}^2 < c^2$.

In such reference frames (designated by ′) we necessarily have from (4) that

$$e' = \frac{M_1 E_2' - M_2 E_1'}{M_1 + M_2} = 0$$

$$t' = t_2' - t_1' = 0, \qquad \text{for any } \tau$$

This implies that

$$t_1' = t_2' = T' = \frac{E'}{Mc^2}\,\tau$$

As $q' = (\vec{q}', 0)$ and $p' = (\vec{p}', 0)$, the relative motion follows the Newton equations of the classical model corresponding formally to the same potential Φ, that is

$$\vec{p}' = \mu \frac{d\vec{q}'}{d\tau}, \qquad \frac{d\vec{p}'}{d\tau} = -\frac{\partial \Phi(|\vec{q}'|)}{\partial \vec{q}'} \tag{6}$$

Consequently there exist reference frames in which the 'geometrical' times t_1' and t_2' of the particles coincide and where the motion is not different from the one given by the corresponding non-relativistic model.

Nevertheless we note the following fact (as from (2)):

$$\begin{aligned} p_1 &= \frac{\mu}{M_2} P - p \\ p_2 &= \frac{\mu}{M_1} P + p \end{aligned} \tag{7}$$

we get

$$\begin{aligned} p_1^2 &= \frac{\mu^2}{M_2^2} P^2 - 2\frac{\mu}{M_2} Pp + p^2 \\ p_2^2 &= \frac{\mu^2}{M_2^2} P^2 - 2\frac{\mu}{M_1} Pp + p^2 \end{aligned} \tag{8}$$

and it is apparent that p_1^2 and p_2^2 are generally not first integrals nor even conserved in a scattering process.

We now examine the scattering problem. Concerning the initial conditions we have

$$\begin{aligned} p_{1\text{in}}^2 &= -M_1^2 c^2 \\ p_{2\text{in}}^2 &= -M_2^2 c^2 \end{aligned} \tag{9}$$

if the particles were not interacting when they were far away from each other [2]. Then we obtain

$$-\frac{1}{c^2}\left(\frac{dq_{i,\text{in}}^2}{d\tau}\right) = \frac{-p_{i,\text{in}}^2}{M_i^2 c^2} = 1, \qquad i = 1, 2$$

and the proper times of the particles coincide with the historical time τ for the initial state, whereas after the scattering process we generally have

$$p_{i,\text{out}}^2 \neq p_{i,\text{in}}^2, \qquad i = 1, 2$$

and the proper times do not coincide with the historical time τ.

Such a state of the particle is called virtual and is in fact unstable. Therefore we postulate the existence of an irreversible process back to the equilibrium during which the particle recovers its characteristic mass M_i.

We then get from the initial conditions according to (9) that

$$K = \frac{p_{1,\text{in}}^2}{2M_1} + \frac{p_{2,\text{in}}^2}{2M_2} = -\frac{Mc^2}{2}$$

and so

$$k = \frac{p^2}{2\mu} + \Phi(|q|) = K - K_0 = \frac{W^2 - M^2c^4}{2Mc^2} \tag{10}$$

where $W = c\sqrt{-P^2}$ is the total energy in a reference frame where $\vec{P} = 0$.

We then have

$$k = \frac{\vec{p}'^2}{2\mu} + \Phi(|\vec{q}'|)$$

Moreover

$$p_{\text{in}}^2 = \vec{p}_{\text{in}}'^2 = \vec{p}_{\text{out}}'^2 = p_{\text{out}}^2 = 2\mu k \tag{11}$$

It is also useful to define a four-vector d which plays the part of an impact parameter such that

$$d' = (\vec{d}', 0) \quad \text{and} \quad \vec{d}'\vec{p}_{\text{in}}' = 0$$

and then

$$dp_{\text{in}} = 0 \tag{12}$$

and according to (5)

$$\tfrac{1}{2}j^{\mu\nu}j_{\mu\nu} = \vec{b}'^2 = \vec{d}'^2\vec{p}_{\text{in}}'^2 = p_{\text{in}}^2 d^2 > 0$$

By a classical calculation it is then easy to obtain the following expression for the angle θ' between the vectors $\vec{p}_{\text{in}}'$ and $\vec{p}_{\text{out}}'$.

$$\theta' = \pi - \left(\frac{j^{\mu\nu}j_{\mu\nu}}{\mu}\right)^{1/2}\int_{\rho_0}^{+\infty}\frac{d\rho/\rho^2}{\left(\dfrac{W^2 - M^2c^4}{2Mc^2} - \Phi(\rho) - \dfrac{j^{\mu\nu}j_{\mu\nu}}{4\mu\rho^2}\right)^{1/2}} \tag{13}$$

where $\rho_0 > 0$ is in general the greatest value of ρ for which the square root under the integral sign vanishes.

Finally as the square of the invariant momentum transfer Δ is expressed according to (7) and (11) as:

$$\begin{aligned}\Delta^2 &= (p_{1,\mathrm{out}} - p_{1,\mathrm{in}})^2 = (p_{2,\mathrm{out}} - p_{2,\mathrm{in}})^2 \\ &= (p_{\mathrm{out}} - p_{\mathrm{in}})^2 = (\vec{p}\,'_{\mathrm{out}} - \vec{p}\,'_{\mathrm{in}})^2 = 4\mu k(1 - \cos\theta') \\ &= 2\mu\,\frac{W^2 - M^2c^4}{Mc^2}(1 - \cos\theta')\end{aligned} \tag{14}$$

we can express Δ, by (13), as a function of the invariants W and $j^{\mu\nu}j_{\mu\nu}$.

For instance in the coulomb case where

$$\Phi(|q|) = \frac{g}{|q|}$$

where g is the coupling constant, we get:

$$\Delta^2 = \frac{8\mu^2 g^2(W^2 - M^2c^4)}{j^{\mu\nu}j_{\mu\nu}(W^2 - M^2c^4) + 2M\mu c^2 g^2}$$

In fact the scattering angle in the reference frame where $\vec{P} = 0$ or in the laboratory frame cannot be expressed in terms of W and Δ without knowing the variations of p_1^2 and p_2^2 due to the scattering process.

K being conserved we can write

$$p_{1,\mathrm{out}}^2 + M_1^2c^2 = 2\varepsilon M_1$$

$$p_{2,\mathrm{out}}^2 + M_2^2c^2 = -2\varepsilon M_2$$

To estimate the coefficient ε we define the four-vector V

$$V^\mu = \tfrac{1}{2} j_{\nu\rho} P_\lambda \varepsilon^{\nu\rho\lambda\mu}$$

so

$$V = (E\vec{b} - \vec{P} \wedge \vec{a}, \vec{P}\vec{b}) \tag{15}$$

Among the reference frames $'$ there is one for which $\vec{P}'$ is perpendicular to $\vec{b}'$. In this particular frame, we have

$$V' = (E'\vec{b}', 0)$$

and in the reference frame where $\vec{P} = 0$.

$$V_r = (W\vec{b}_r, 0)$$

so

$$V^\mu V_\mu = E'^2\vec{b}'^2 = \tfrac{1}{2}E'^2 j^{\mu\nu}j_{\mu\nu} = W^2\vec{b}_r^2 \tag{16}$$

With the help of (7) we have

$$\varepsilon = \frac{p_{1,\mathrm{out}}^2 - p_{1,\mathrm{in}}^2}{2M_1} = \frac{1}{M}(Pp_{\mathrm{in}} - Pp_{\mathrm{out}})$$

According to the initial conditions (9) and the relations (8) we find

$$Pp_{\text{in}} = \vec{P}'\vec{p}'_{\text{in}} = (M_2 - M_1)\,\frac{W^2 - M^2c^4}{2Mc^2} \tag{17}$$

and as $\vec{P}'$, $\vec{p}'_{\text{in}}$ and $\vec{p}'_{\text{out}}$ belong to the same plane we can write

$$Pp_{\text{out}} = \vec{P}'\vec{p}'_{\text{out}} = \vec{P}'\vec{p}'_{\text{in}}\left(\cos\theta' \pm \sin\theta'\left(\frac{\vec{P}'^2\vec{p}'^2_{\text{in}}}{(\vec{P}'\vec{p}'_{\text{in}})^2} - 1\right)^{1/2}\right)$$

where the $\pm$ must be chosen according to the sign of $\overleftrightarrow{b}'(\vec{P}' \wedge \vec{p}'_{\text{in}})$.

Finally using (14), (16) and (17) we get

$$\begin{aligned}\varepsilon = {} & \frac{M_2 - M_1}{4\mu M}\,\Delta^2 \\ & \pm \frac{\Delta}{Mc}\left((W^2 - M^2c^4 - \Delta^2 Mc^2/4\mu)\left(\frac{2V^\mu V_\mu - M^2c^4 j^{\mu\nu}j_{\mu\nu}}{j^{\mu\nu}j_{\mu\nu}(W^2 - M^2c^4)} - \frac{M}{4\mu}\right)\right)^{1/2}\end{aligned} \tag{18}$$

where the $\pm$ must be chosen according to the sign of $(M_2 - M_1)\sin\theta'\overleftrightarrow{b}'(\vec{P}' \wedge \vec{p}'_{\text{in}})$.

We note that the scattering process is characterized by the invariants

$$P^\mu P_\mu, \qquad j^{\mu\nu}j_{\mu\nu} \qquad \text{and} \qquad V^\mu V_\mu$$

that is to say by

$$W, \qquad \overleftrightarrow{b}_r^2 - \vec{a}_r^2/c^2 \qquad \text{and} \qquad \overleftrightarrow{b}_r^2$$

We now consider the bound states, assuming as before $j^{\mu\nu}j_{\mu\nu} > 0$. Again according to the fact that the relative motion takes place in a plane of the q^μ-coordinate space, the study of the motion is reduced to that of the corresponding classical model.

For the initial conditions in such a case we impose

$$P^2 = -M^2c^2 \tag{19}$$

So the proper time of the center of mass coincides with the historical time τ.

Therefore in the case of the Kepler problem (Coulomb potential with $g < 0$) the orbits are ellipses whose foci coincide with the center of mass in the reference frames $'$.

In a different reference frame we also have ellipses but their foci do not coincide with the center of mass.

It is important to note that p_1^2 and p_2^2 are first integrals if and only if the orbit is circular in some reference frame $'$. In this case we necessarily have $\vec{P}'$ parallel to $\overleftrightarrow{b}'$ that is to say that $\vec{P}'$ is perpendicular to the orbital plane. This implies that there is a particular reference frame $'$ in which

$$t'_1 = t'_2 = T' = \tau$$

Consequently it is easy to see that an elliptic eccentricity for the orbit in a reference frame $'$ corresponds to a periodic fluctuation of the times t_1 and t_2 about T [3] in the reference frame in which $\vec{P} = 0$, because from (2)

$$t_1 = T - \frac{M_1}{M}\,t$$

$$t_2 = T + \frac{M_2}{M}\,t$$

Concerning the case where $j^{\mu\nu}j_{\mu\nu} \leqslant 0$ the relative motion still takes place in a plane of the q^{μ}-coordinate space but this plane is no longer space-like. In fact there exist particular reference frames (designated by ′) characterized by pure Lorentz transformations given by:

$$v = c^2 \frac{\vec{a} \wedge \vec{b}}{\vec{a}^2} + \lambda \frac{\vec{a}}{|\vec{a}|}, \text{ such that } \vec{v}^2 < c^2$$

in which the relative motion is such that $\vec{b}' = 0$.

It is not possible to give results for general cases. We have only studied the case of the Coulomb potential (given previously).

We have obtained the following results:

Firstly if $g < 0$ any initial condition (such that $j^{\mu\nu}j_{\mu\nu} \leqslant 0$) provides a relative trajectory by which the light cone is reached in a finite time and in this case the model for the potential Φ is of course unrealistic since physically it must be bounded by some mass and cannot be infinite for any state.

Secondly if $g > 0$ we obtain the same results unless

$$0 < k < \frac{-\mu g^2}{2j^{\mu\nu}j_{\mu\nu}},$$

in which case we have a scattering process and $p_{\text{in}}^2 = 2\mu k > 0$. We can choose the reference frame ′ such that the initial conditions can be written as:

$$p'_{\text{in}} = (\vec{p}'_{\text{in}}, 0)$$
$$d' = (0, d'^4)$$

So the four-vector d' which plays the part of impact parameter is purely time-like. The previous inequality then leads to

$$0 < k < \frac{g}{2\sqrt{2}cd'^4},$$

since

$$\tfrac{1}{2}j^{\mu\nu}j_{\mu\nu} = -\vec{a}^2/c^2 = -c^2(d'^4)^2\vec{p}'^2_{\text{in}} = d^2 p_{\text{in}}^2$$

To conclude, in spite of the apparent difficulties due to the metric of the space-time, our model describes well what we consider to be a relativistic two-body system and as we have seen, it is possible, for each Newtonian system with a potential $\Phi(|\vec{q}\,|)$, to construct a completely covariant relativistic model describing the same type of motion.

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