On the foundations of relativity

Autor(en): **Aaberge, Terje**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **48 (1975)**

Heft 2

PDF erstellt am: **30.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-114668

Helvetica Physica Acta
Vol. 48, 1975. Birkhäuser Verlag Basel

On the Foundations of Relativity

by **Terje Aaberge**[1])

Département de Physique Théorique, Université de Genève, 32, Bd. d'Yvoy, CH—1211 Genève 4, Suisse

(16. XII. 74)

Abstract. We sketch a language in which to discuss the notion of relativity in general in connection with space-time coordinatization, and apply this language to interpret the theory of special relativity. We then argue that this theory does not give the only possible representation of 'Einstein Relativity', and to show this we give a (non-linear) realization of the Lorentz group on $\mathbb{R}^4$ and use this to arrive at the usual interpretation of the μ-meson experiment. Finally, we briefly introduced the notion of passivity in a scheme which can serve as a basis for the construction of a Hamiltonian dynamics for Einstein relativistic particles.

1. The Principle of Relativity and its Interpretation

> One of the basic principles of physics is the principle of relativity, which says:
> the results of a certain well-defined experiment do not depend on the frame of reference in which it is performed.

Stated in this way, the principle does not give much more than an implicit definition of frame of reference, and to interpret it in a given context one must complement the preceding statement with postulates defining the context and the role of relativity in it.

Thus, starting with the notion of frame of reference as primitive, let us denote the set of frames of references $\{\lambda\}$, and let us postulate that for a given frame of reference λ:

P_1) time is represented by $(E)_\lambda$, the one-dimensional Euclidean space;

P_2) physical space is represented by $(E_3)_\lambda$, the three-dimensional Euclidean space;

then as a comment on the principle of relativity we must also postulate that:

P_3) P_1 and P_2 are satisfied for any frame of reference $\lambda \in \{\lambda\}$.

Accordingly, we have to consider a family of space-times $(E \times E_3)_\lambda$ $(=(E)_\lambda \times (E_3)_\lambda)$, one for each frame of reference λ.

A coordinatization of $(E \times E_3)_\lambda$, being a labelling of the points of $(E \times E_3)_\lambda$ by the points of $\mathbb{R}^4$, is represented by an affine bijection of $\mathbb{R}^4$ onto $(E \times E_3)_\lambda$. For a given frame of reference λ with space-time $(E \times E_3)_\lambda$ one must distinguish between two kinds of such coordinizations. The one which is performed by an apparatus in the frame λ, denoted $O_{\lambda\lambda}$,

$$\mathbb{R}^4 \xrightarrow{O_{\lambda\lambda}} (E \times E)_\lambda;$$

1) This paper contains results from the author's Ph.D. thesis. The research has been partially supported by the Fonds National Suisse.

and those being performed by the length-measuring device, and possibly the clock of any other frame λ', denoted $O_{\lambda'\lambda}$,

$$\mathbb{R}^4 \xrightarrow{O_{\lambda'\lambda}} (E \times E_3)_\lambda.$$

With regard to the first kind of coordinatizations we postulate that:

P$_4$) for any two frames of reference:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "O_{\lambda\lambda}"] \arrow[d, "I"'] & (E \times E_3)_\lambda \\
\mathbb{R}^4 \arrow[r, "O_{\lambda'\lambda'}"] & (E \times E_3)_{\lambda'};
\end{tikzcd}$$

where I denotes a canonical identification. This postulate is meant to say that in all frames of reference one uses the same kind of clocks and length-measuring devices, and according to the same prescription.

With respect to a given frame of reference λ with space-time $(E \times E_3)_\lambda$, we define the notion of relativity transformation by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^4 \arrow[drr, "O_{\lambda''\lambda}"] \arrow[dr, "O^{\lambda}_{\lambda''}"'] \arrow[dd, "O^{\lambda''}_{\lambda'}"'] & & \\
& \mathbb{R}^4 \arrow[r, "O_{\lambda\lambda}"] & (E \times E_3)_\lambda \\
\mathbb{R}^4 \arrow[ur, "O^{\lambda}_{\lambda'}"] \arrow[urr, "O_{\lambda'\lambda}"'] & &
\end{tikzcd}$$

Its interpretation follows from the notion of coordinatization.

It is important to realize that in this definition we have chosen a frame of reference and refer to the space-time of this frame only, thus exhibiting what seems important in relativity, namely:

i) the equivalence of space-times of different frames of reference;

ii) the asymmetry involved in a relativity transformation. By the fact that, for the space-time $(E \times E_3)_\lambda$ of a frame of reference λ there exist a 'preferred' coordinatization $O_{\lambda\lambda}$ made by apparatuses in the λ-frame. This is what Einstein [1] refers to as a 'stationary system'.

We are now in a position to define what is a frame of reference in this context. For this we postulate that:

P$_6$) for any given frame of reference λ any other frame of reference is represented by a line in $(E \times E_3)_\lambda$, its world-line; and in particular, the world-line of λ is the time axis in the coordinatization $O_{\lambda\lambda}$.

To complete the list of postulates we also state the following:

P$_7$) the configuration of world-lines does not depend on the frame of reference λ to which the corresponding world-lines are referred.

P$_7$ is an extension of our formulation of the principle of relativity. It is a statement about the relation between the frames of reference, and together with the other postulates (especially P$_4$ and P$_6$) it affirms that:

the relativity transformation $O^{\lambda}_{\lambda'}$ maps the world-line of λ onto the world-line of λ' etc. whatever frame of reference λ'' (i.e. space-time $((E \times E_3)_{\lambda''})$) the world-lines are referred to.

It is thus precisely this proposition which, when the coordinatizations $O_{\lambda\lambda}$, $O_{\lambda'\lambda}$ etc. are defined, permits one to determine the structure of the relativity group modulo the translations of $\mathbb{R}^4$.

2. Einstein Relativity

The postulates P_1–P_7 are common to the notion of relativity and relativity transformation in Galilei as well as Einstein physics, and they have been given for the purpose of making a more precise interpretation of relativity. To obtain either Galilei or Einstein relativity one must correspondingly add some postulates to define coordinatizations of the kind $O_{\lambda\lambda}$, $O_{\lambda'\lambda}$ etc. of the space-time $(E \times E_3)_\lambda$.

We will not do this; rather, we assume that some such postulates have been given and that they have conducted us to Einstein relativity in the sense of special relativity; that is, $O^{\lambda}_{\lambda'}$ is a representation of the inhomogeneous Lorentz group on $\mathbb{R}^4$, which restricted to the homogeneous sub-group (the Lorentz group) satisfy the relation

$$\Lambda^{\alpha}{}_{\beta}\, g_{\alpha\gamma}\, \Lambda^{\gamma}{}_{\delta} = g_{\beta\delta},$$

for,

$$g_{\alpha\beta} = \begin{cases} -c^2 & \alpha = \beta = 0 \\ 1 & \alpha = \beta = 1, 2, 3 \\ 0 & \alpha \neq \beta, \end{cases}$$

where Λ denotes a representation of the Lorentz group on $\mathbb{R}^4$ (we use the usual summation convention); i.e., for a special Lorentz transformation the explicit form of Λ is $(\gamma = (1 - (u^2/c^2))^{-1/2})$

$$(\Lambda(\vec{u})^{\alpha}{}_{\beta}) = \begin{bmatrix} \gamma & \gamma\dfrac{u_1}{c^2} & \gamma\dfrac{u_2}{c^2} & \gamma\dfrac{u_3}{c^2} \\ \gamma u_1 & 1 + \dfrac{\gamma^2}{\gamma+1}\dfrac{u_1^2}{c^2} & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_1 u_2}{c^2} & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_1 u_3}{c^2} \\ \gamma u_2 & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_1 u_2}{c^2} & 1 + \dfrac{\gamma^2}{\gamma+1}\dfrac{u_2^2}{c^2} & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_2 u_3}{c^2} \\ \gamma u_3 & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_1 u_3}{c^2} & \dfrac{\gamma^2}{\gamma+1}\dfrac{u_2 u_3}{c^2} & 1 + \dfrac{\gamma^2}{\gamma+1}\dfrac{u_3^2}{c^2} \end{bmatrix},$$

and for the rotations

$$(\Lambda(\vec{\theta})^{\alpha}{}_{\beta}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R(\vec{\theta})^{i}{}_{j} & \\ 0 & & & \end{bmatrix},$$

where R is the usual representation of the rotation group $SO(3)$ on $\mathbb{R}^3$.

To make an interpretation of the Lorentz transformations, we will look at the following diagram:

$$\begin{tikzcd}
\mathbb{R}^4 \arrow[dr, "O_{\lambda'\lambda}"] & \\
& (E \times E_3)_\lambda \\
\mathbb{R}^4 \arrow[uu, "O^{\lambda}_{\lambda'}"'] \arrow[dd, "O^{\lambda'}_{\lambda}"] \arrow[ur, "O_{\lambda\lambda}"] \arrow[dr, "O_{\lambda'\lambda'}"'] & \\
& (E \times E_3)_{\lambda'}. \\
\mathbb{R}^4 \arrow[ur, "O_{\lambda\lambda'}"'] &
\end{tikzcd}$$

According to the rules of the art of drawing (two-dimensional) coordinate systems in Minkowski-space, the upper half of the diagram corresponds to:

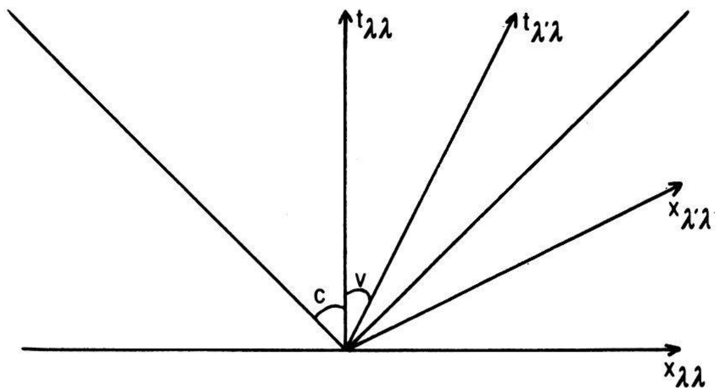

Figure 1

and the lower half to:

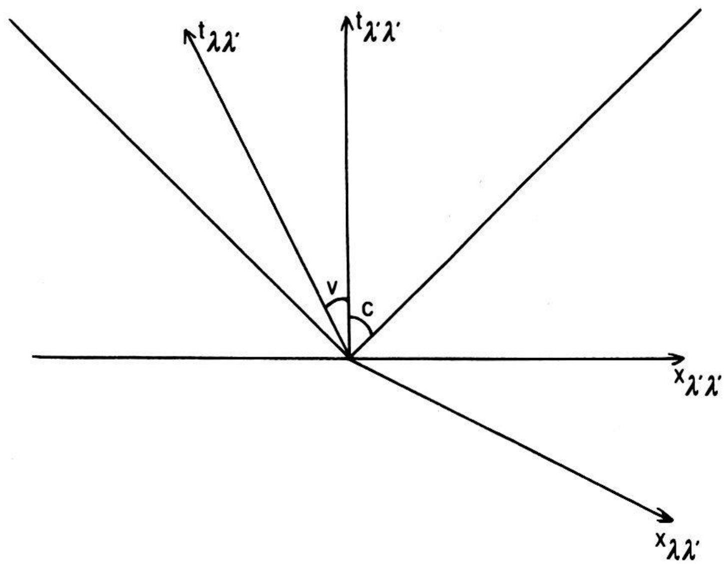

Figure 2

when the frames of reference λ and λ' move with a relative velocity v and their world-lines intersect at the origin.

The interpretation of the figures follows from the interpretation of coordinatization and relativity-transformation:

> $t_{\lambda\lambda}$ and $x_{\lambda\lambda}$ are the time and space coordinates of a given point in $(E \times E_3)_\lambda$, measured by apparatuses in the frame λ;

and,

> $t_{\lambda'\lambda}$ and $x_{\lambda'\lambda}$ the coordinates of the same point in $(E \times E_3)_\lambda$ measured by apparatuses in λ';

similarly:

> $t_{\lambda'\lambda'}$ and $x_{\lambda'\lambda'}$ are the time and space coordinates of a given point in $(E \times E_3)_{\lambda'}$, measured by apparatuses in the frame λ';

and,

> $t_{\lambda\lambda'}$ and $x_{\lambda\lambda'}$ the coordinates of the same point in $(E \times E_3)_{\lambda'}$ measured by apparatuses in λ.

In particular, according to postulate P_4, the measures of time satisfy

$$t_{\lambda\lambda} = t_{\lambda'\lambda'} \quad \text{and} \quad t_{\lambda\lambda'} = t_{\lambda'\lambda};$$

t now denoting a unit of duration. This is the assumption on which the usual interpretation of the results of the μ-meson experiment is based [2].

It follows from the interpretation of relativity in general and from the preceding discussion of special relativity, that there exist a possibility of redefining the coordinatizations of the kind $O_{\lambda'\lambda}$ as follows:

> $O_{\lambda'\lambda}$ is the coordinatization of $(E \times E_3)_\lambda$ obtained by using a clock in the λ-frame (always), but a length-measuring device in the λ'-frame.

One may ask for the 'representation' $O^{\lambda}_{\lambda'}$ of the Lorentz group on $\mathbb{R}^4$ corresponding to this interpretation, and for this purpose we will define the four-velocity space and three-velocity space at a given point $\mathbf{p}$ of the configuration space of special relativity.

Let λ be a given frame and $\mathbf{p}$ a point in its space $(E \times E_3)_\lambda$ through which the time-axis $t_{\lambda\lambda}$ passes, and consider the subset of frames of reference whose world-lines pass through $\mathbf{p}$. The world-line of, say, λ' is then naturally parametrized by

$$\{(\gamma_{\vec{v}}, \gamma_{\vec{v}}\vec{v})\, t_{\lambda\lambda} \mid t_{\lambda\lambda} \in \mathbb{R}\}, \quad \text{for} \quad \gamma_{\vec{v}} = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-1/2}$$

where $\vec{v}$ is the relative (three-) velocity between λ and λ', i.e.,

$$(\gamma_{\vec{v}}, \gamma_{\vec{v}}\vec{v})\, t_{\lambda\lambda} = \Lambda(\vec{v})^{\alpha}{}_{\beta}(1, \vec{0})^{\beta}\, t_{\lambda\lambda},$$

where $\Lambda(\vec{u})^{\alpha}{}_{\beta}$ represent $O^{\lambda}_{\lambda'}$. Accordingly, every world-line of the considered sub-set of frames of reference is completely specified relative to λ by a four-velocity w^{μ}, satisfying

$$w^{02} c^2 - \vec{w}^2 = c^2;$$

or a three-velocity $\vec{v}$. The set of all four-velocities constitutes a hyperbolic hypersurface in a four-dimensional space, a surface of transitivity for the Lorentz group, i.e.:

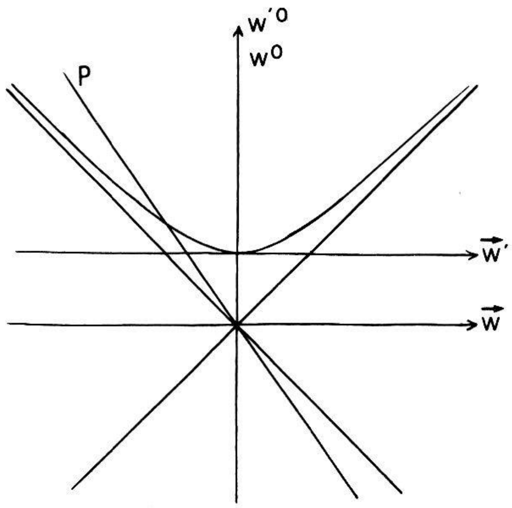

Figure 3

The relation between the four-velocity space and the Lobachewskian space of three-velocities is a hyperbolic projection P (see Fig. 3),

$$w^{\mu} \xmapsto{P} \vec{v} = \frac{\vec{w}}{w^0},$$

of the hyperbolic surface into the hyper-plane through its vertex. This Lobachewskian space is completely characterized by saying that its group of motions is the Lorentz group [3].

Now, we are on the level of postulate P_7, and can start to construct the desired 'representation' $O^{\lambda}_{\lambda'}$. For this purpose we construct a four-velocity space by translating the coordinate axis in the four-dimensional space into which the hyperbolic surface is embedded by (see Fig. 3)

$$w^0 \xmapsto{T} w'^0 = w^0 - 1, \quad \vec{w} \xmapsto{T} \vec{w}' = \vec{w};$$

then, in this new coordinate system, the hyperbolic surface is given by

$$(w'^0 + 1)^2 c^2 - \vec{w}'^2 = c^2,$$

and the hyperbolic projection by

$$w'^{\alpha} \xmapsto{P'} \vec{v} = \frac{\vec{w}'}{w'^0 + 1},$$

i.e.

$$w'^{\alpha} = (w'^0, \vec{w}') = (\gamma_{\vec{v}} - 1, \gamma_{\vec{v}} \vec{v}).$$

Furthermore, the Lorentz group acts on the hyperbolic surface in the following manner:

$$w'^{\alpha} \xmapsto{AG} \Lambda(\vec{u}, \vec{\theta})^{\alpha}{}_{\beta}\, w'^{\beta} + u'^{\alpha},$$

for Λ as already defined, and

$$u'^{\alpha} = (\gamma_{\vec{u}} - 1, \gamma_{\vec{u}}\vec{u}).$$

Λ_G leaves the hyperbolic surface,

$$(w'^0 + 1)^2 c^2 - \vec{w}'^2 = c^2,$$

invariant, and is related to the representation Λ by

$$\Lambda_G = T\Lambda T^{-1}.$$

To obtain a 'representation' $O^{\lambda}_{\lambda'}$ on the space-time, we first observe that

$$\vec{v} = \frac{\vec{w}'}{w'^0 + 1} = \vec{w}' - \frac{\vec{w}'}{w'^0 + 1} w'^0 = \vec{w}' - \vec{v}w'^0.$$

Since we assume the time-measure t to be invariant, the world-line of the λ'-frame in the $O_{\lambda\lambda}$ coordinatization must be written[2])

$$\{(\tau, \vec{v}\tau) | \tau \in \mathbb{R}\} = \left\{ \left(\tau, \vec{w}'\tau - \frac{\vec{w}'}{w^0 + 1} \tau \right) | \tau \in \mathbb{R} \right\},$$

and we can define a mapping of this line onto a line in a five-dimensional space S by

$$\{(\tau, \vec{v}\tau) | \tau \in \mathbb{R}\} \xrightarrow{\alpha} \{(\tau, w'^{\alpha}\tau) | \tau \in \mathbb{R}\},$$

with respect to a given coordinatization of S.

Denoting a general element of S by (τ, q^{α}), the invariance of τ makes it natural to choose the following action of the Lorentz group on S:

$$(\tau, q^{\alpha}) \mapsto (\tau, \Lambda(\vec{u}, \vec{\theta})^{\alpha}{}_{\beta} q^{\beta} + u'^{\alpha}\tau).$$

More specifically, from the definition of the embedding α, we find that $O^{\lambda}_{\lambda'}$ for a special Lorentz transformation is given by

$$(\tau, \vec{q}) \mapsto \left(\tau, \vec{q}' - \frac{\vec{u}}{c} q'^0 \right),$$

for

$$\left(\frac{q'^0}{c'}, \vec{q}' \right)^{\alpha} = \Lambda(\vec{u})^{\alpha}{}_{\beta}(0, \vec{q})^{\beta} + u^{\alpha}\tau;$$

or, explicitly,

$$\tau \mapsto \tau' = \tau$$

$$\vec{q} \mapsto \vec{q}' = \vec{q} - \frac{\gamma}{\gamma + 1} \frac{\vec{q} \cdot \vec{u}}{c^2} \vec{u} + \vec{u}\tau,$$

---

2) Subsequently we will denote $t_{\lambda\lambda}$ by τ.

i.e.

$$\vec{q}\,'_{\parallel} = \vec{q}_{\parallel}\left(1-\frac{\vec{u}^2}{c^2}\right)^{1/2} + \vec{u}\tau \quad \text{and} \quad \vec{q}\,'_{\perp} = \vec{q}_{\perp},$$

and we note that this gives a non-linear realization of the Lorentz group.

We would like to remark that when one constructs a Hamilton-dynamics on 'this' basis, for classical and quantal particles respectively, one can define observables

$$\vec{q}_s = \vec{q} - \frac{\vec{p}}{p^0 + mc}\, q^0$$

for the classical particle, and

$$\vec{q}_s = \vec{q} - \frac{1}{2}\left[\frac{\vec{p}}{p^0 + mc}, q^0\right]_+$$

for the quantal particle, corresponding to the 'inverse' of α, i.e. $\vec{q} - (\vec{u}/c)q^0$. It turns out [4] that in the quantal case, this observable corresponds to the Newton–Wigner position observable [5]. Its interpretation should be clear from the preceding; it is the observable describing the position of the particle in three-dimensional physical space.

3. Remark on the μ-Meson Experiment

The most important experiment said to conform with the theory of special relativity is probably the μ-meson experiment [6] the result of which one usually interprets in terms of time-dilation, i.e. that the time scale $t_{\lambda'\lambda}$ of the rest frame λ' of a μ-meson moving with a velocity $\vec{u}$ relative to the frame λ of the laboratory (as seen from the λ-frame) is dilated by a factor $\gamma = (1 - (\vec{u}^2/c^2))^{-1/2}$ in relation to the time scale $t_{\lambda\lambda}$ of the laboratory frame.

The results of this experiment and the interpretation thereof are also compatible with our realization of Einstein relativity, although the 'numerical' results are not so 'easily' arrived at as in the case of special relativity. In fact, in special relativity this result is obtained directly by performing a Lorentz transformation on $t_{\lambda\lambda}$, while in our case we must derive the result, since time is invariant.

As the preceding calculations show, the scale $x_{\lambda'\lambda}$ for distance measurements in the rest frame λ' of the μ-meson is contracted in the direction of its motion relative to the frame of the laboratory λ. Thus, if $q_{\lambda\lambda}$ is the distance between the two counters in λ as measured by apparatuses in this frame, then the distance is $q_{\lambda'\lambda}$ when measured by apparatuses in the λ' frame, for

$$q_{\lambda'\lambda} = q_{\lambda\lambda}\left(1-\frac{\vec{u}^2}{c^2}\right)^{1/2}.$$

Now

$$\Delta\tau = \frac{q_{\lambda\lambda}}{u}$$

is the duration of time (in λ seen from λ) needed for the μ-meson to cover the distance between the two counters; equivalently, applying postulate P_6, we find that

$$\Delta\tau' = \frac{q_{\lambda\lambda}}{u} = \frac{q_{\lambda\lambda}}{u}\left(1-\frac{u^2}{c^2}\right)^{1/2} = \Delta\tau\left(1-\frac{u^2}{c^2}\right)^{1/2}$$

is the corresponding duration of time in λ measured by a clock in λ'. Thus we get the same result as in the theory of special relativity:

$$\Delta\tau = \gamma\Delta\tau' \text{ }^{3)}.$$

Similarly, calculations on the simple experiments and thought experiment usually presented in elementary texts on special relativity give the same results in our theory. This does not mean that our theory is completely equivalent to special relativity, and that one can translate from one to the other in all conceivable cases. It is only when we erect more complex structures on the basis given by the two frameworks, by constructing field-theories and Hamiltonian particle dynamics that the differences appear.

4. The Passive Point of View

The point of view under which we have discussed relativity so far is by definition the active point of view. We define the passive point of view to be associated with the following definition of $O_{\lambda'\lambda}$:

it is the coordinatization obtained by using a length-measuring device in λ' and a clock in λ, however, always referring to the origin in λ.

Applying the idea of passivity, we can transcribe the new kind of Lorentz transformation to read:

$$q^\alpha \mapsto \Lambda(\vec{u},\vec{\theta})^\alpha{}_\beta q^\beta$$

$$w'^\alpha \mapsto \Lambda(\vec{u},\vec{\theta})^\alpha{}_\beta w'^\beta + u'^\beta.$$

Equivalently, the passivity can be expressed in terms of the language of the theory of special relativity, by defining coordinatizations of Minkowski-space with respect to an origin fixed to the frame of reference, and thus moving along its world-line.

This last remark gives a correspondence between the usual theory and ours.

References

[1] A. Einstein, Annln Phys. *17* (1905).
[2] D. H. Frisch and J. H. Smith, Am. J. Phys. *31*, 342 (1963).
[3] J. M. Gelfand, R. A. Minlos and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, Part II, Chap. I, Sect. 1, §6 (Pergamon Press, Oxford 1963).
[4] T. Aaberge, Helv. Phys. Acta *48*, 173 (1975).
[5] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. *21*, 400 (1949).
[6] See Ref. [2].

3) One should be aware of the role played by postulate P_4 in the usual case as well as in ours. Applied to this experiment it says that with respect to its instantaneous rest-frame, the μ-meson always has the same half-time, irrespective of its velocity.