From the asymptotic condition to the cross-section

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From the Asymptotic Condition to the Cross-section

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Abstract. A relation is established in the framework of algebraic scattering theory between the time dependent formalism and the cross-section: the formula concerning 'scattering into cones' is shown to hold essentially as a consequence of the asymptotic condition.

1. Introduction

In the algebraic formulation of non-relativistic scattering theory, one expresses the asymptotic condition by imposing convergence of the time evolution of a certain algebra of observables [1]–[5]. The algebra $\mathscr{A}$ to be considered must contain a complete set of commuting observables characterizing the particles participating in the scattering process, i.e., individual momenta and spins. This requirement implies that $\mathscr{A}$ has an abelian commutant $\mathscr{A}' \subseteq \mathscr{A}$. A suitable choice for $\mathscr{A}$ is the von Neumann algebra generated by the momenta or the von Neumann algebra of all constants of the free motion. Given $\mathscr{A}$ and the group of total evolution V_t, the system $(\mathscr{A}, V_t)$ satisfies the asymptotic condition if

$$\text{s-}\lim_{t\to\pm\infty} V_t^* A V_t \psi = A_\pm \psi \tag{1}$$

exists for all $A \in \mathscr{A}$ on the set of scattering states ψ.

Under suitable conditions (namely that the correspondences between $\mathscr{A}$ and the asymptotic algebra $\mathscr{A}_\pm$ are one-to-one and the image of a maximal abelian algebra is still maximal abelian for single channel scattering), one may deduce the existence of two classes of isometries $\Omega_\pm$ such that

$$A_\pm = \Omega_\pm A \Omega_\pm^*. \tag{2}$$

$\Omega_\pm$ are determined only up to multiplication from the right by a unitary operator belonging to $\mathscr{A}'$. It is then possible to define a class of scattering operators S by the usual formula

$$S = \Omega_+^* \Omega_-. \tag{3}$$

As a consequence of the indeterminateness of the wave operators, two S-operators in the class can differ by a unitary factor in $\mathscr{A}'$.

In order to establish a bridge between the time dependent formulation of scattering and the cross-section, one needs to relate the theoretically calculated S-operator to the quantities actually measured in a scattering experiment. Experimental setups are concerned with the probability of finding the scattered particle located in a cone C in space, whose apex coincides with the position of the scattering center. If $V_t \Omega_- \phi$ is the scattering state at time t, this probability is $\int_C |(V_t \Omega_- \phi)(\mathbf{x})|^2 d^3 x$.

On the other hand we can compute from the asymptotic condition the probability for the outgoing particle to have its momentum in the same cone C in momentum space. With (1), (2) and (3) this quantity is

$$\lim_{t \to \infty} (\Omega_- \phi, V_t^* F_c V_t \Omega_- \phi) = \int_C |(\widetilde{S\phi})(\mathbf{k})|^2 d^3 k$$

where F_C is the projection operator on C, and $\tilde{\phi}(\mathbf{k})$ is the Fourier transform of $\phi(\mathbf{x})$. The desired link will be established if one can prove the equality of these two quantities when t goes to infinity [6, 7, 8].

$$\lim_{t \to \infty} \int_C |(V_t \Omega_- \phi)(\mathbf{x})|^2 d^3 x = \int_C |(\widetilde{S\phi})(\mathbf{k})|^2 d^3 k. \tag{4}$$

It is clear that the right-hand side is independent of the choice of the representative in the equivalence class of S-operators, so that this indeterminateness is irrelevant for the computation of physical quantities.

We want to show in this work that the validity of (4) is essentially a consequence of the asymptotic condition (1). For this, one must study the asymptotic time behavior of the scattering state $V_t \Omega_- \phi$. It is not postulated that the motion becomes free in the usual sense as $t \to +\infty$, but rather its nature will be investigated as a consequence of the asymptotic condition for each scattering system $(\mathscr{A}, V_t)$.

An asymptotic evolution is a time dependent family of closed operators T_t, each of them affiliated to $\mathscr{A}'$, such that

$$\Omega h = \text{s-}\lim_{t \to \infty} V_t^* T_t h. \tag{5}$$

h belongs to a domain $\mathscr{D}$ dense in $\mathscr{H}$ and invariant under $\mathscr{A}$.[1]) Only the asymptotic properties of T_t are of importance, and we shall call equivalent two asymptotic evolutions T_t^1 and T_t^2 which satisfy

$$\lim_{t \to \infty} \|(T_t^1 - T_t^2) h\| = 0 \qquad \text{for } h \text{ belonging to a common dense domain}$$

Two equivalent asymptotic evolutions define the same wave operator Ω. The asymptotic evolution T_t should be distinguished from the free motion U_t generated by the purely kinematical part of the energy. They agree in the case of short range forces, but for long range interaction, the T_t incorporate their residual effect at large distance from the scattering center.

It can be proved that a scattering system $(\mathscr{A}, V_t)$ always admits a class of asymptotic evolutions T_t [1, 2, 5]. The asymptotic condition (1) also puts strong restrictions on the time behavior of T_t. These restrictions, supplemented by convenient assumptions on T_t, are precisely those under which the formula (4) can be derived. The supplementary hypothesis concerns the possibility of choosing a sufficiently regular representative in the class of asymptotic evolutions (i.e., unitarity, with continuity and differentiability properties in t). Without further specification of the interaction, these properties cannot be deduced from the abstract formulation of the asymptotic condition. However, they are verified for all the potentials for which the asymptotic condition is known to hold.

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[1]) In the following we shall only consider the limit $t \to +\infty$. Similar statements hold for $t \to -\infty$.

2. General Properties of Asymptotic Evolutions

The wave operator intertwines two groups V_t and W_t where W_t is defined by the equation

$$W_t = \Omega^* V_t \Omega. \tag{6}$$

W_t is a strongly continuous unitary group of operators in $\mathscr{A}'$ which differs in general from the asymptotic evolution T_t. For short and long range potentials, W_t is found to be identical with the group generated by the kinetic energy $H_0 = |\mathbf{p}|^2/2m$. We shall assume in the following that this is the case, although we could allow more general W_t taking into account a renormalization of the free energy.

The relation between W_t and the asymptotic evolution T_t is described by the following lemma.

Lemma 1. *Let T_t be an asymptotic evolution. Then for any real τ,*

$$\lim_{t\to\infty} \|(T_{t+\tau} - W_\tau T_t) h\| \to 0, \qquad h \in \mathscr{D}.$$

Proof. Since V_t is unitary, one has

$$\lim_{t\to\infty} \|(T_{t+\tau} - W_\tau T_t) h\| = \lim_{t\to\infty} \|(V_t^* T_{t+\tau} - V_t^* T_t W_\tau) h\|.$$

But for $h \in \mathscr{D}$, the asymptotic condition (5) and the intertwining relation implies

$$\text{s-lim}\, V_t^* T_{t+\tau} h = \text{s-lim}\, V_{t-\tau}^* T_t h = V_\tau \,\text{s-lim}\, V_t^* T_t h$$

$$= V_\tau \Omega h = \Omega W_\tau h = \text{s-lim}\, V_t^* T_t W_\tau h. \qquad \blacksquare$$

It we denote by $\tilde{W}_t \equiv W_t^* T_t$ the effective difference between T_t and W_t we obtain

$$\lim_{t\to\infty} \|(\tilde{W}_{t+\tau} - \tilde{W}_t) h\| = 0. \tag{7}$$

A family of vectors ϕ_t having the property (7) cannot be a too rapidly increasing function of t, as is shown in Lemma 2.

Lemma 2. *Let ϕ_t be a uniformly strongly continuous family of vectors for $t \in [t_0, \infty)$ such that*

$$\lim_{t\to\infty} \|\phi_{t+\tau} - \phi_t\| = 0 \;\forall \tau \in [0, \tau_0]$$

then

$$\lim_{t\to\infty} \frac{\|\phi_t\|}{t} = 0.$$

Proof. For a fixed τ, one can find a number K such that $\|\phi_{n\tau+\tau} - \phi_{n\tau}\| < \epsilon$ for $n \geqslant K$. Then we can write

$$\frac{1}{N\tau} \|\phi_{N\tau}\| = \frac{1}{N\tau} \|\phi_0 + \sum_{n=0}^{N-1} (\phi_{(n+1)\tau} - \phi_{n\tau})\|$$

$$\leqslant \frac{1}{N\tau} \left\| \phi_0 + \sum_{n=0}^{K} (\phi_{(n+1)\tau} - \phi_{n\tau}) \right\| + \frac{1}{N\tau} \sum_{n=K}^{N-1} \|\phi_{(n+1)\tau} - \phi_{n\tau}\|.$$

The last term is less than ϵ/τ, so that we obtain $\lim_{N\to\infty} 1/N\tau\|\phi_{N\tau}\| = 0$. For an arbitrary t we have with $|N\tau - t| < \tau$

$$\frac{\|\phi_t\|}{t} = \left\| \frac{1}{t}(\phi_t - \phi_{N\tau}) + \left(1 + \frac{N\tau - t}{t}\right)\frac{\phi_{N\tau}}{N\tau} \right\|$$
$$\leqslant \frac{1}{t}\|\phi_t - \phi_{N\tau}\| + \left(1 + \frac{\tau}{t}\right)\frac{\|\phi_{N\tau}\|}{N\tau}.$$

The uniform continuity of ϕ_t implies that $\|\phi_t - \phi_{N\tau}\|$ is uniformly bounded in t. Therefore $\|\phi_t\|/t$ converges to zero as t goes to infinity. ∎

The same result could be obtained under the weaker condition that ϕ_t is strongly continuous and $\lim_{t\to\infty}\|\phi_{t+\tau} - \phi_t\| = 0$ for all $\tau > 0$ ([12], Chapter VIII, 7, Ex. 5).

It is always possible to select a particular representative of the equivalence class of asymptotic evolutions which has continuity and differentiability properties in t. The construction is done in the following way [2]. Let e be a cyclic vector for $\mathscr{A}$ chosen in the domain of H_0, and let $\mathscr{D}$ be the dense set of vectors of the form $h = Ae$, $A \in \mathscr{A}$. We define

$$T'_t h = ACV_t \Omega e. \tag{8}$$

C is the projection on the subspace spanned by the set $\{\mathscr{A}' e\}$.

Lemma 3. *The asymptotic evolution defined by Equation* (8) *is uniformly continuous and uniformly differentiable on* $\mathscr{D}$, $t \in [0, \infty)$.

Proof. The uniform continuity of $T'_t h$ follows from that of the group V_t. If $e \in \mathscr{D}_{H_0}$, then $\Omega e \in \mathscr{D}_H$, and we have

$$\left\| \left(\frac{T'_{t+\tau} - T'_t}{\tau}\right) h + iACV_t H\Omega e \right\| \leqslant \|AC\| \left\| \left(i\frac{V_\tau - 1}{\tau} - H\right)\Omega e \right\| < \epsilon$$

for $\tau < \delta$ in view of the differentiability of the group V_t. ∎

The particular asymptotic evolution T'_t is not known to be unitary, nor is it even known whether it is uniformly bounded in t. However, in all explicitly treated cases it is possible to find a unitary family T_t in the equivalence class. A necessary and sufficient condition for this has been given in [5]. We shall therefore assume that we can find a unitary uniformly differentiable asymptotic evolution. Such an evolution is of the form

$$T_t = \exp[-i(H_0 t + F_t)] \tag{9}$$

where F_t is a family of self-adjoint operators affiliated to $\mathscr{A}'$. We shall also assume that T_t gives rise to a differential equation for the asymptotic motion of the scattering state:

$$i\frac{dT_t}{dt}h = \left(H_0 + \frac{dF_t}{dt}\right)T_t h, \qquad h \in \mathscr{D}. \tag{10}$$

Proposition 1. Let T_t be a unitary asymptotic evolution, uniformly differentiable on $\mathscr{D}$ for $t \in [t_0, \infty)$, then

$$\lim_{t\to\infty}\left\|\frac{dF_t}{dt}h\right\| = \lim_{t\to\infty}\|(F_{t+\tau}-F_t)\,h\| = \lim_{t\to\infty}\frac{1}{t}\|F_t h\| = 0$$

Proof.

$$\left\|\frac{dF_t}{dt}h\right\| = \left\|\left(i\frac{dT_t}{dt} - H_0 T_t\right)h\right\|$$

$$= \left\|i\left(\frac{dT_t}{dt} - \frac{T_{t+\tau}-T_t}{\tau}\right)h + i\left(\frac{T_{t+\tau}-W_\tau T_t}{\tau}\right)h + T_t\left(i\,\frac{W_\tau - 1}{\tau} - H_0\right)h\right\|$$

$$\leqslant \left\|\left(\frac{dT_t}{dt} - \frac{T_{t+\tau}-T_t}{\tau}\right)h\right\| + \frac{1}{\tau}\|(T_{t+\tau}-W_\tau T_t)\,h\| + \left\|\left(i\frac{W_\tau - 1}{\tau} - H_0\right)h\right\|.$$

For a sufficiently small fixed τ, the first and the third terms can be made less than ϵ for any $t \in [t_0, \infty)$ and the second term converges to zero as t goes to infinity by Lemma 1. The two other limits follow from the identities

$$\operatorname*{s-lim}_{t\to\infty}(F_{t+\tau}-F_t)\,h = \operatorname*{s-lim}_{t\to\infty}\int_t^{t+\tau} dt'\,\frac{dF_{t'}}{dt'}h = 0$$

$$\operatorname*{s-lim}_{t\to\infty}\frac{1}{t}F_t h = \operatorname*{s-lim}_{t\to\infty}\left(\frac{1}{t}F_{t_0}h + \frac{1}{t}\int_{t_0}^{t} dt'\,\frac{dF_{t'}}{dt'}h\right) = 0. \qquad \blacksquare$$

One sees that the 'distortion' F_t which appears in $\widetilde{W}_t = e^{iF_t}$ also has the characteristics described in Lemma 2: it cannot increase faster than t.

3. Scattering into Cones

For sake of definiteness, we choose $\mathscr{A}$ to be the maximal abelian algebra generated by the momentum $\mathbf{p}$ for single channel scattering. The 'distortion' F_t is some real function $F_t(\mathbf{p})$. In order to study the trajectory $T_t^* Q_i T_t$ of the particle under its asymptotic motion, it is necessary to introduce some more detailed assumptions on the function $F_t(\mathbf{p})$. A suitable set of properties is

i) There exists a dense set $\mathscr{C}$ on which the p_i are essentially self-adjoint and which is a common domain for all F_t, $t > 0$.
ii) The vectors $F_t\phi$, $\phi \in \mathscr{C}$, belong to the domain of Q_iQ_j, $i, j = 1, 2, 3$ (that is, $F_t(\mathbf{p})$ is at least twice differentiable in $\mathbf{p}$).
iii) The vectors $Q_i F_t \phi$ and $Q_i Q_j F_t \phi$ are strongly uniformly continuous for $t \in [t_0, \infty)$. (11)

Proposition 2. Assume that T_t is a unitary uniformly differentiable asymptotic evolution with properties (10) and (11). Then

$$\operatorname*{s-lim}_{t\to\infty}\left(\frac{1}{t}T_t^* Q_i T_t - \frac{p_i}{m}\right)\phi = 0, \qquad i = 1, 2, 3, \qquad \phi \in \mathscr{C}. \tag{12}$$

and the formula (4) holds.

Proof. It follows from (11), (i) and (ii) that $T_t\phi$ belongs to the domain of Q_i and a simple calculation yields

$$\frac{1}{t}T_t^* Q_k T_t\phi - \frac{p_k}{m}\phi = \frac{Q_k\phi}{t} + \frac{iF_tQ_k\phi}{t} - \frac{iQ_kF_t\phi}{t}.$$

The first term converges obviously to zero, as well as the second by Proposition 1. Let us show that it is also the case for the third one. One obtains by Schwartz inequality

$$\|Q_k(F_{t+\tau} - F_t)\phi\|^2 \leqslant \|(F_{t+\tau} - F_t)\phi\|\,\|Q_k^2(F_{t+\tau} - F_t)\phi\|.$$

(11), (ii) and (iii) insure that $\|Q_k^2(F_{t+\tau} - F_t)\phi\|$ is uniformly bounded in t for fixed τ. Thus the family of vectors $Q_kF_t\phi$ verifies the hypothesis of Lemma 2 and consequently

$$\lim_{t\to\infty}\frac{\|Q_kF_t\phi\|}{t} = 0.$$

This proves equation (12). The convergence of the family of self-adjoint operators $(1/t)\,T_t^*Q_iT_t$ to p_i/m on a core of p_i entails the convergence of the respective spectral families ([13], Chapter VIII, Th. 1.5 and 1.15). Arguing as Jauch, Lavine and Newton [8] we conclude that

$$\lim_{t\to\infty}\int_C |(T_t\phi)(\mathbf{x})|^2\,d^3x = \int_C |\tilde{\phi}(\mathbf{k})|^2\,d^3k \tag{13}$$

for a general class of cones C. The proof of this fact is precisely the content of their main theorem. Then the formula of scattering into cones (4) follows from (13) and the asymptotic condition in the form (5) (see Lemma 3 of [7]). ■

The formula (4) remains true if the generator of the group W_t differs from H_0, provided that it is of the type considered in [8] (i.e., a function of $|\mathbf{p}|$ with positive derivative).

Unitary asymptotic evolutions have been explicitly constructed for the Coulomb potential by Dollard [9] and for more general long range forces in [1, 10, 11]. We summarize the results in quoting the theorems proved in [11].

Let the long range part $V(\mathbf{x})$ of the potential and all its partial derivatives D_x^k of order k, $k \leqslant 3$ satisfy the estimate

$$|D_x^k V(\mathbf{x})| \leqslant c(1 + |\mathbf{x}|)^{-\alpha-k} \tag{14}$$

with $\alpha > \frac{1}{2}$. Then the function $F_t(\mathbf{p})$ is given by

$$F_t(\mathbf{p}) = \int_0^t V\left(\frac{\mathbf{p}t'}{m}\right)dt'.$$

Let us indicate how all the regularity conditions used in propositions 1 and 2 are verified.

(1) $F_t(\mathbf{p})$ is clearly differentiable, and since $V(\mathbf{x})$ is a bounded function $F_t(\mathbf{p})$ satisfies a Lifschitz condition

$$|F_{t+\tau}(\mathbf{p}) - F_t(\mathbf{p})| \leqslant M\tau. \tag{15}$$

From this it is easy to see that F_t is strongly differentiable on $\mathscr{H}$. Then

$$\lim_{t\to\infty} \left\| \frac{dF_t}{dt} f \right\|^2 = \lim_{t\to\infty} \int \left| V\left(\frac{\mathbf{p}t}{m}\right)\right|^2 |f(\mathbf{p})|^2 d^3p = 0$$

so that the conclusions of proposition 1 hold.[2])

(2) With the help of inequality (14) one proves that the p derivatives of $F_t(\mathbf{p})$ satisfy also a Lifschitz condition in t:

$$|D_p^k F_{t+\tau}(\mathbf{p}) - D_p^k F_t(\mathbf{p})| \leqslant \int_t^{t+\tau} \left| D_p^k V\left(\frac{\mathbf{p}t'}{m}\right)\right| dt' \leqslant c\frac{\tau}{|\mathbf{p}|^k}. \tag{16}$$

If $\mathscr{C}$ is chosen as the set of functions of the form $|\mathbf{p}|^2\phi(\mathbf{p})$ where $\phi(\mathbf{p})$ belongs to the space of test functions of Schwartz, (i) and (ii) are true. One deduces from (15) and (16) that

$$\|(Q_i F_{t+\tau} - Q_i F_t)\phi\| \leqslant M_1\tau, \qquad \phi\in\mathscr{C}$$
$$\|(Q_i Q_j F_{t+\tau} - Q_i Q_j F_t)\phi\| \leqslant M_2\tau, \qquad \phi\in\mathscr{C}$$

from which (iii) follows.

In the case where $\frac{1}{2} \geqslant \alpha > 1/n$ and the partial derivatives of the potential obey the estimates (14) up to order $n+2$, $F_t(\mathbf{p})$ must be defined as [11]

$$F_t(\mathbf{p}) = \int_0^t h_n(\mathbf{p}, t')\, dt$$

where $h_1(\mathbf{p},t) = 0$ and the $h_k(\mathbf{p},t)$, $1 \leqslant k \leqslant n$ are determined by the successive iteration

$$h_k(\mathbf{p},t) = V\left(\frac{\mathbf{p}t}{m} + \nabla_p \int_0^t h_{k-1}(\mathbf{p},t')\, dt'\right).$$

The inequalities

$$|D_p^k h_n(\mathbf{p},t)| \leqslant c\left| D_p^k V\left(\frac{\mathbf{p}t}{m}\right)\right| \qquad \text{for } 0 \leqslant k \leqslant 3$$

enable us to conclude that the preceding results remain true in the case $0 < \alpha \leqslant \frac{1}{2}$.

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[2]) The uniform differentiability of the asymptotic evolution T_t can also be proved.

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