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Definition of Green's Functions for Dilute Fermi Gases

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Abstract. The infinite volume limit is shown to exist for Green's functions of a large class of dilute Fermi gases. In particular, this gives meaning to the time evolution of the corresponding infinite systems.

I started struggling to understand statistical mechanics about ten years ago in Zurich, with the kind help of M. Fierz and R. Jost. It is nice to remember these discussions which Fierz insisted to have in German—after a few minutes German was replaced by Schwytzer-Dütsch. Few mathematically solid results were then known about thermodynamic limits, correlation functions, reduced density matrices, etc. Now, 'rigorous results' have become an industry but I hope that M. Fierz will find some pleasure in looking at this note dedicated to him on his sixtieth birthday.

It was shown in a recent note [6] that Ginibre's work on the reduced density matrices of quantum gases [3], [4] can be used to discuss the existence and analyticity of Green's functions. Here we sharpen the results of [6] in the case of Fermi systems. In particular the time correlation functions are seen to be well defined in the limit of an infinite volume for a dilute Fermi gas.

Green's functions are defined by

$$G_\Lambda^{\boldsymbol{\phi}}(\boldsymbol{\zeta}) = Z^{-1}\,\mathrm{Tr}(A_1\, e^{-(\zeta_2-\zeta_1)H_\Lambda} A_2 \ldots e^{-(\zeta_m-\zeta_{m-1})H_\Lambda} A_m\, e^{-(\beta+\zeta_1-\zeta_m)H_\Lambda})$$

where H_Λ is the Hamiltonian for a bounded region Λ of $\mathbb{R}^\nu$, $Z = \mathrm{Tr}\, e^{-\beta H_\Lambda}$, A_k is either $\int dx\, \phi_k(x) a(x)$ or $\int dx\, \phi_k(x) a^*(x)$, and $\phi_k \in L^2(\mathbb{R}^\nu)$[1]. We have used the notation $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m)$, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)$ and assumed that $\boldsymbol{\zeta} \in \bar{\mathscr{D}}$,

$$\bar{\mathscr{D}} = \{\boldsymbol{\zeta}: \mathrm{Re}\,\zeta_1 \leqslant \cdots \leqslant \mathrm{Re}\,\zeta_m \leqslant \mathrm{Re}\,\zeta_1 + \beta\}$$

For a system of particles interacting through a suitable pair potential Φ,[2] and for small activity, the operator $e^{-\lambda H_\Lambda}$, with $\lambda > 0$, may be defined in terms of Wiener

[1]) More generally one could take

$$A_k = \int dx_1 \ldots dx_p\, dy_1 \ldots dy_q\, \phi_k(x_1, \ldots, x_p, y_1, \ldots, y_q) a^*(x_1) \ldots a^*(x_p)\, a(y_1) \ldots a(y_q)$$

where $\phi_k \in L^2(\mathbb{R}^{\nu(p+q)})$, see [6].

[2]) Ginibre's conditions [4] are (A), (B) of the theorem below and

(C) $\int_{|x|>c} |\Phi(x)|\, dx < +\infty$ *for some* $C > 0$.

We shall replace (C) by the stronger requirement

(C′) $\Phi \in L^1(\mathbb{R}^\nu) \cap L^2(\mathbb{R}^\nu)$.

integrals and is of trace class (see Ginibre [1]). The operators $A_k e^{-\lambda H_\Lambda}$ can also be expressed in terms of Wiener integrals and are of trace class.

When λ is complex and $\operatorname{Re}\lambda > 0$, $e^{-\lambda H_\Lambda}$ is defined and analytic, therefore G_Λ^Φ is an analytic function of the complex variables $\zeta_k = \beta_k - it_k$ in the domain

$$\mathscr{D} = \{\boldsymbol{\zeta} : \beta_1 < \cdots < \beta_m < \beta_1 + \beta\}$$

If $\boldsymbol{\zeta} \in \overline{\mathscr{D}}$ and $t_1 = \cdots = t_n$, Ginibre's analysis shows that G_Λ^Φ tends to a limit when $\Lambda \to \infty$.[3]) From this the following result can be deduced (see [6]):

There exists a function G^Φ analytic in $\mathscr{D}$ and such that

$$\lim_{\Lambda\to\infty} G_\Lambda^\Phi(\boldsymbol{\zeta}) = G^\Phi(\boldsymbol{\zeta}) \tag{1}$$

uniformly on compacts with respect to $\boldsymbol{\zeta} \in \mathscr{D}$.

The theorem below shows that $G_\Lambda^\Phi(\boldsymbol{\zeta})$ has a limit G^Φ when $\Lambda \to \infty$ and $\boldsymbol{\zeta}$ is in the closure $\overline{\mathscr{D}}$ of $\mathscr{D}$. This defines infinite volume Green's functions for real times and not just $\zeta \in \mathscr{D}$.

Theorem. *Let $\Phi : \mathbb{R}^\nu \to \mathbb{R}$ satisfy the following conditions*

(A) *Φ is even ($\Phi(x) = \Phi(-x)$) and continuous for $x \neq 0$*
(B) *Φ is stable*
(C′) $\int |\Phi(x)|\, dx < +\infty$, $\int |\Phi(x)|^2\, dx < +\infty$

For a Fermi system with pair potential Φ at small activity, G^Φ extends to a bounded continuous function on $\overline{\mathscr{D}}$ such that

$$\lim_{\Lambda\to\infty} G_\Lambda^\Phi(\boldsymbol{\zeta}) = G^\Phi(\boldsymbol{\zeta})$$

uniformly on the compacts of $\overline{\mathscr{D}}$.

We decompose the proof in several steps

Lemma 1. *Let $\mathscr{M}_n$ be defined by*

$$\mathscr{M}_n = \left\{\boldsymbol{\zeta} \in \overline{\mathscr{D}} : \beta_3 - \beta_1 \geqslant \frac{\beta}{n}, \beta_4 - \beta_2 \geqslant \frac{\beta}{n}, \ldots \right.$$
$$\left. \ldots, \beta_m - \beta_{m-2} \geqslant \frac{\beta}{n}; \beta + \beta_1 - \beta_{m-1} \geqslant \frac{\beta}{n}, \beta + \beta_2 - \beta_m \geqslant \frac{\beta}{n} \right\}$$

For sufficiently large Λ, G_Λ^Φ is bounded on $\mathscr{M}_n$ uniformly with respect to $\boldsymbol{\zeta}$ and Λ; if ϕ_k is of class C^2 with compact support, the same is true of $\partial G_\Lambda^\Phi / \partial \zeta_k$. Furthermore G_Λ^Φ tends to a limit G^Φ uniformly on the compacts of $\mathscr{M}_n$ when $\Lambda \to \infty$.

If $\boldsymbol{\zeta} \in \mathscr{M}_n$, G_Λ^Φ can be expressed in terms of operators

$$e^{[it_k - (1/4n)\beta]H_\Lambda} A_k(\phi_k) e^{-it_k H_\Lambda}, \quad e^{it_k H_\Lambda} A_k(\phi_k) e^{-[it_k + (1/4n)\beta]H_\Lambda},$$

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[3]) The bounded open $\Lambda \subset \mathbb{R}^\nu$ form a directed set, when ordered by inclusion. We let $\lim G_\Lambda$ when $\Lambda \to \infty$ be by definition the limit of the directed family (G_Λ).

and $e^{-\lambda H_\Lambda}$ with $0 < \lambda < \beta$. Using Hölder's inequality [4]) we find an upper bound for $|G^\Phi_\Lambda|$ in terms of the expressions

$$Z^{-1}\,\mathrm{Tr}[(e^{-(1/4n)\beta H_\Lambda}A_k(\phi_k)A_k(\phi_k)^*\,e^{-(1/4n)\beta H_\Lambda})^n]$$

and

$$Z^{-1}\,\mathrm{Tr}[(e^{-(1/4n)\beta H_\Lambda}A_k(\phi_k)^*\,A_k(\phi_k)e^{-(1/4n)\beta H_\Lambda})^n]$$

These expressions are known to have a limit when $\Lambda \to \infty$. This proves the uniform boundedness of $|G^\Phi_\Lambda(\boldsymbol{\zeta})|$.

The derivative $\partial G^\Phi_\Lambda/\partial\zeta_k$ is obtained by replacing $A_k(\phi_k)$ by $[A_k(\phi_k), H_\Lambda]$ in the expression of G^Φ_Λ. One is thus brought to considering the expression

$$Z^{-1}\,\mathrm{Tr}[(e^{-(1/4n)\beta H_\Lambda}[A_k(\phi_k), H_\Lambda][A_k(\phi_k), H_\Lambda]^*\,e^{-(1/4n)\beta H_\Lambda})^n] \tag{2}$$

or a similar one with $[A_k(\phi_k), H_\Lambda]$ replaced by $[A_k(\phi_k), H_\Lambda]^*$. Since ϕ_k is of class C^2 with compact support, the commutator of $A_k(\phi_k)$ with the kinetic energy part of H_Λ is again of the form $A(\phi)$. The potential energy part of H_Λ is

$$\sim \int dx \int dy\, \Phi(x-y)a^*(x)a^*(y)a(x)a(y)$$

and its commutator with $A_k(\phi_k)$ is

$$\sim \int dx \int dy\, \phi_k(x)\Phi(x-y)a^*(x)a^*(y)a(y)$$

or

$$\sim \int dx \int dy\, \phi_k(x)\Phi(x-y)a^*(y)a(x)a(y).$$

We insert these expressions in

$$[A_k(\phi_k), H_\Lambda][A_k(\phi_k), H_\Lambda]^*$$

and use the anticommutation relations to put the annihilation operators a to the right and the creation operators a^* to the left (Wick ordering). A number of terms are thus obtained which are conveniently described by diagrams; they are integrals of Wick ordered products $a^*(x_1)\ldots a(x_r)$ multiplied by continuous functions $\phi(x_i)$, and the pair potential $\Phi(x_i - x_j)$. The pair potential appears as factor 0, 1, or 2 times. If $\Phi(x_i - x_j)$ appears there also appears a factor $\phi(x_i)$ or $\phi(x_j)$; for each variable x_i which does not appear in a factor $\Phi(x_j - x_i)$ there is a factor $\phi(x_i)$. Inserting now in (2) we obtain a sum of terms which are integrals of reduced density matrices multiplied by factors $\phi(x_k)$ and $\Phi(x_k - x_j)$ as described above. Using the condition (C′) of the theorem, and the fact that the reduced density matrices are bounded functions, uniformly in Λ [3],[5]) we obtain a bound on (2) which is independent of Λ. This gives the desired boundedness of $\partial G^\Phi_\Lambda/\partial\zeta_k$.

The C^2 functions with compact support are dense in L^2. Therefore we may assume that $\phi_1, \ldots, \phi_m$ are C^2 with compact support in proving that G^Φ_Λ tends to a limit G^Φ uniformly on the compacts of $\mathscr{M}_n$ when $\Lambda \to \infty$. Since the G^Φ_Λ and their derivatives $\partial G^\Phi_\Lambda/\partial\zeta_k$ are uniformly bounded, this results from $G^\Phi_\Lambda(\zeta) \to G^\Phi(\zeta)$ for $\zeta \in \mathscr{D}$ (see (1)).

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[4]) See [2] Lemma XI, 9–20, p. 1105.

[5]) An easy extension of Ginibre's results is actually needed at this point.

Lemma 2. *Let Φ be given and Λ_0 chosen sufficiently large. We assume that* $1 \leqslant l \leqslant m$, *that K is compact in $\mathbb{R}$ and that $\zeta_{l+1}, \ldots, \zeta_m$ are fixed such that*

$$0 < \operatorname{Re} \zeta_{l+1} < \cdots < \operatorname{Re} \zeta_m < \frac{\beta}{2}$$

Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m) - G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_{l-1}, \zeta_l, \ldots, \zeta_m)| < \epsilon \tag{3}$$

whenever $\Lambda \supset \Lambda_0$, and θ_l, ζ_l satisfy $\theta_l \in K$, $|\zeta_l + i\theta_l| < \delta$, $0 < \operatorname{Re} \zeta_l < \operatorname{Re} \zeta_{l+1}$.

Furthermore, $G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m)$ has a limit $G_{(l)}^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m)$ when $\Lambda \to \infty$, uniformly for $(\theta_1, \ldots, \theta_l) \in K^l$.

We have

$$\begin{aligned}
&|G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m) - G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_{l-1}, \zeta_l, \ldots, \zeta_m)| \\
&= |Z^{-1} \operatorname{Tr}[A_1 e^{i(\theta_2 - \theta_1)H_\Lambda} \ldots A_{l-1} \\
&\quad \times (e^{i(\theta_l - \theta_{l-1})H_\Lambda} A_l e^{-(\zeta_{l+1} + i\theta_l)H_\Lambda} - e^{-(\zeta_l + i\theta_{l+1})H_\Lambda} A_l e^{-(\zeta_{l+1} - \zeta_l)H_\Lambda}) \\
&\quad \times A_{l+1} \ldots e^{-(\zeta_m - \zeta_{m-1})H_\Lambda} A_m e^{-(\beta - i\theta_1 - \zeta_m)H_\Lambda}]| \\
&\leqslant XY
\end{aligned}$$

where

$$\begin{aligned}
X^2 &= Z^{-1} \operatorname{Tr}(e^{-\beta H_\Lambda/2} A_1 e^{i(\theta_2 - \theta_1)H_\Lambda} \ldots A_{l-1} A_{l-1}^* \ldots e^{-i(\theta_2 - \theta_1)H_\Lambda} A_1^* e^{-\beta H_\Lambda/2}) \\
&\leqslant (\|A_1\| \ldots \|A_{l-1}\|)^2
\end{aligned}$$

$$Y^2 = [F(\zeta_l, \zeta_l) - F(\zeta_l, -i\theta_l) - F(-i\theta_l, \zeta_l) + F(-i\theta_l, -i\theta_l)]$$

$$\begin{aligned}
F(u, v) = Z^{-1} \operatorname{Tr}[&e^{-(\beta/2 - i\theta_1 - \zeta_m)^* H_\Lambda} A_m^* \ldots A_{l+1}^* e^{-(\zeta_{l+1} - u)^* H_\Lambda} A_l^* \\
&\times e^{(-u^* - v)H_\Lambda} A_l e^{-(\zeta_{l+1} - v)H_\Lambda} A_{l+1} \ldots A_m e^{-(\beta/2 - i\theta_1 - \zeta_m)H_\Lambda}]
\end{aligned}$$

and (3) results from Lemma 1.

We prove now the convergence of $G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m)$ when $\Lambda \to \infty$. For $l = 1$ this is again a consequence of Lemma 1. For the general case we use induction on l. By the compactness of K, given $\delta > 0$, there exist $\zeta_l^{(1)}, \ldots, \zeta_l^{(N)}$ with the following properties

a) $0 < \operatorname{Re} \zeta^{(j)} < \operatorname{Re} \zeta_{l+1}$,
b) for each $\theta_l \in K$ there is a j such that

$$|\zeta_l^{(j)} + i\theta_l| < \delta.$$

By the first part of the lemma we have thus

$$|G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m) - G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_{l-1}, \zeta_l^{(j)}, \zeta_l, \ldots, \zeta_m)| < \epsilon.$$

On the other hand the induction assumption implies that, for $(\theta_1, \ldots, \theta_{l-1}) \in K^{l-1}$ and Λ sufficiently large, we have

$$|G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_{l-1}, \zeta_l^{(j)}, \ldots, \zeta_m) - G_{(l-1)}^\Phi(-i\theta_1, \ldots, -i\theta_{l-1}, \zeta_l^{(j)}, \ldots, \zeta_m)| < \epsilon.$$

The uniform convergence of $G_\Lambda^\Phi(-i\theta_1, \ldots, -i\theta_l, \zeta_{l+1}, \ldots, \zeta_m)$ follows.

Proof of the Theorem

Since the functions G_Λ^ϕ are analytic in $\mathscr{D}$ and bounded continuous in $\bar{\mathscr{D}}$, it is easily seen that uniform convergence on the compacts of the distinguished boundary $\mathscr{S}$ implies uniform convergence on the compacts of $\bar{\mathscr{D}}$. The distinguished boundary is defined by

$$\mathscr{S} = \bigcup_{l=1}^{m} \mathscr{S}_l$$

$$\mathscr{S}_l = \{\boldsymbol{\zeta} : \operatorname{Re}\zeta_1 = \cdots = \operatorname{Re}\zeta_l,\ \operatorname{Re}\zeta_{l+1} = \cdots = \operatorname{Re}\zeta_m = \operatorname{Re}\zeta_1 + \beta\}.$$

Therefore by Lemma 2 we have uniform convergence on the compacts of $\mathscr{S}$ when $\Lambda \to \infty$, and the theorem is proved.

Remark on time evolution

It is known that for certain quantum lattice systems, time evolution can be described by automorphisms of a 'standard' C*-algebra (see [5]). In this note we prove only the existence of Green's functions (a similar result is known for a large class of lattice systems [7]). It is probable that time evolution cannot be described here by automorphisms of the C*-algebra of canonical anticommutation relations, and it is unclear if it is given by automorphisms of its weak closure. For further considerations on this problem see [1].

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