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On the Quantum Mechanical N-Body Problem[1])

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Abstract. Systems of a finite number of nonrelativistic particles are studied in the framework of time-independent quantum scattering theory in the approach of Faddeev. For a non-empty class of 2-body potentials, we shall prove the unitarity of the S-matrix and a singularity structure of resolvent kernels and scattering amplitudes in the physical region, which is qualitatively the same as in perturbation theory.

§0. Introduction

The investigation of quantum mechanical many particle systems has received much attention in the recent past. The most developed framework is non-relativistic wave mechanics which, apart from its importance for a correct description of vast domains of natural phenomena, provides a very interesting testing ground for studying general properties of quantum scattering amplitudes.

Two goals seem to be within reach in a rigorous treatment of the Schrödinger equation for a Hamilton operator with short range forces: the proof of the unitarity of the S-matrix and of maximal regularity. Maximal regularity is a concept of qualitative dynamics (a field full of refreshing new results in classical mechanics [22]): the N-body resolvent kernels, in particular the scattering amplitudes, are expected to be as regular as in perturbation theory augmented by some obvious "kinematical" singularities connected with bound states. The smoothness of these amplitudes in p-space ("maximal analyticity" in the best of all worlds), except for rescattering and threshold singularities, would strongly support our space-time picture of wave mechanics as expressed by the asymptotic condition and the cluster decomposition properties of the scattering amplitudes. The use of quantum mechanical amplitudes, which are only "kernels" of isometric operators in Hilbert space, is greatly facilitated if maximal regularity holds, e.g. in the expression for cross sections or virial coefficients and, more fundamentally, for the most natural proof of unitarity: we know from formal scattering theory that non-relativistic quantum mechanics of finitely many particles satisfies asymptotic completeness, once some mild regularity properties of the resolvent kernels in the continuous spectrum allow the interchange of certain limits.

The central idea in the pioneering work of Faddeev [3] on the 3-body problem is based on maximal regularity: highly connected products of 2-body amplitudes become uniformly smooth in the physical region and allow to treat the singular limit in

[1]) Elaboration of lectures given at the 5th Polish Winter School of Theoretical Physics at Karpacz, February 1968.

the scattering integral equations. Our investigation will systematically develop this technique and prove the qualitative relevance of perturbation theory for multiparticle processes and thus unitarity.

§ 1 Scattering Theory and F-Y Equations

This section will start with time-dependent scattering theory, mainly for illustrating of the kinematics and dynamics of multichannel systems. Then we shall introduce the FADDEEV-YAKUBOVSKY (F-Y) equations for the N-body resolvent and establish the connection between time-independent and time-dependent scattering theory.

We study quantum mechanical N-body systems with a Hamiltonian $H = H^0 + V$ in $\mathcal{H} = L^2(R^{3N})$ of the form

$$H^0 = \sum_{i=1}^{N} \frac{p_i^2}{2\, m_i}\,, \qquad V = \sum_{1 \leqslant i < j \leqslant N} V_{ij}\,(x_i - x_j) \tag{1.1}$$

V_{ij} is real and in $L^2(R^3)$, and therefore H is selfadjoint on the domain of H^0 [1]. All particles are distinguishable, i.e. there is no non-trivial permutation of the particle coordinates which leaves H invariant. The discussion of Fermi systems, in particular a systematic low density expansion for infinite Fermi systems at $T \geq 0$, will be treated in a separate paper.

The N particles are indexed by 1, 2, ... N. Let $\mathfrak{a} \in \{1, \ldots N\}$ denote any subsystem with a Hamiltonian

$$H_{\mathfrak{a}} = \overline{H}{}^0_{\mathfrak{a}} + \hat{H}_{\mathfrak{a}}\,, \qquad \hat{H}_{\mathfrak{a}} = \hat{H}^0_{\mathfrak{a}} + V_{\mathfrak{a}}\,, \qquad V_{\mathfrak{a}} = \sum_{i,j \in \mathfrak{a}} V_{ij}\,. \tag{1.2}$$

$\overline{H}{}^0_{\mathfrak{a}}$ is the free Hamiltonian of the center-of-mass (CM) movement of $\mathfrak{a}$ and $\hat{H}^0_{\mathfrak{a}}$ describes the free movement in the relative coordinates within $\mathfrak{a}$ (see Appendix A). If $\mathfrak{a}$ contains more than one element, let $\mathfrak{F}_{\mathfrak{a}} = \{\psi^n_{\mathfrak{a}}\}$ be an orthonormal basis of the discrete spectrum of $\hat{H}_{\mathfrak{a}}$, where $\hat{H}_{\mathfrak{a}}\, \psi^n_{\mathfrak{a}} = E^n_{\mathfrak{a}}\, \psi^n_{\mathfrak{a}}$. Otherwise we retain one $\psi_{\mathfrak{a}} = 1$ with $E_{\mathfrak{a}} = 0$. A channel A_i of i fragments is a set

$$A_i = \{\psi^{n_j}_{\mathfrak{a}_j} \in \mathfrak{F}_{\mathfrak{a}_j}\,, \qquad 1 \leq j \leq i\} \tag{1.3}$$

where $a_i = a(A_i) = \{\mathfrak{a}_1, \ldots \mathfrak{a}_i\}$ is any partition of $\{1, \ldots N\}$ into i non-empty sets. The channel wave function is in compact notation

$$\psi_{A_i} = \bigotimes_{j=1}^{i} \psi^{n_j}_{\mathfrak{a}_j}\,. \tag{1.4}$$

A channel Hamiltonian $H^0_{A_i}$ is defined on $\mathcal{H}$ by

$$H^0_{A_i} = \sum_{j=1}^{i} \left(\overline{H}{}^0_{\mathfrak{a}_j} + E^{n_j}_{\mathfrak{a}_j}\right) \tag{1.5}$$

and is in p-space a multiplication operator by $E_{A_i}(p)$ (see App. A). There is only one channel A_N with N fragments and energy $E(p)$. For every partition a_i there is a tensor product decomposition

$$\mathcal{H} = \overline{\mathcal{H}}_{a_i} \otimes \hat{\mathcal{H}}_{a_i} \tag{1.6}$$

effected by introducing instead of $p_1, \ldots p_N$ the CM coordinates $\bar{p}(\mathfrak{a}_j)$ and relative coordinates $\hat{p}_i(\mathfrak{a}_j)$. Whenever there is no confusion we omit the number i of fragments in a channel $A = A_i$ or a partition $a = a_i$.

The main result of time-dependent scattering theory is [2]:

Theorem 1.1: For $V_{ij} \in L^2(R^3) + L^p(R^3)$, $2 \leq p < 3$, and all channels A

$$\underset{t \to \pm\infty}{s\text{-lim}} \exp(i\, H\, t) \exp(-\, i\, H_A^0 t) = \Omega_A^{\pm} \tag{1.7}$$

exists on $\mathcal{H}_A = \overline{\mathcal{H}}_A \otimes \psi_A$. For all $A \neq B$, $\mathcal{H}_B^{ex} \perp \mathcal{H}_A^{ex}$, where $\mathcal{H}_A^{ex} = \Omega_A^{ex}\, \mathcal{H}_A$, ex $= \pm$.

The states $\Omega_A^{ex} f_A \otimes \psi_A$ have physical interpretation as incoming $(-)$ or outgoing $(+)$ scattering states. When propagated under $\exp(-\, i\, H\, t)$, they have the free wave packet $\exp(-\, i\, H_A^0 t)\, f_A \otimes \psi_A$ with CM wave function $f_A \in \overline{\mathcal{H}}_A$ as "asymptote" in the L^2 norm for $t \to \pm\, \infty$. The Ω_A^{ex} are intertwining operators,

$$\exp(-\, i\, H\, t)\, \Omega_A^{ex} = \Omega_A^{ex} \exp(-\, i\, H_A^0\, t)\,, \tag{1.8}$$

and H is on $\mathcal{H}_A^{ex}$ unitarily equivalent to H_A^0. Thus the spectrum $\sigma(\hat{H})$ of $\hat{H}$ contains the continuous spectrum of all $\hat{H}_A^0 = H_A^0 - \bar{H}^0$:

$$\sigma(\hat{H}) \supset \left[\min_{A_i,\, i \geqslant 2} \sum_{j=1}^{i} E_{\mathfrak{a}_j}^{n_j}, \infty \right) \equiv [E^c, \infty)\,. \tag{1.9}$$

The S-matrix is defined as isometric mapping from $\mathcal{H}^+ = \bigoplus_A \mathcal{H}_A^+$ onto $\mathcal{H}^- = \bigoplus_A \mathcal{H}_A^-$ by

$$S\, \Omega_A^+ f_A \otimes \psi_A = \Omega_A^- f_A \otimes \psi_A\,. \tag{1.10}$$

We are interested in proving asymptotic completeness, i.e.

$$\mathcal{H} = \mathcal{H}^+ = \mathcal{H}^- \tag{1.11}$$

which insures the unitarity of S on $\mathcal{H}$.

The scattering amplitudes $(\Omega_A^+ f_A \otimes \psi_A, \Omega_B^- g_B \otimes \psi_B)$ are tempered distributions. We shall show that there exist integral operators which contain all information about the asymptotic observables of the system. From now on we shall stay entirely in the CM system of $\{1, \ldots N\}$ and write H for $\hat{H}$, etc. Of central importance is the resolvent $R(z) = (z - H)^{-1}$ which is holomorphic for $z \notin \sigma(H)$. Let $R_0(z) = (z - H^0)^{-1}$. The resolvent equation

$$R(z) = R_0(z) + R_0(z)\, V\, R(z) \tag{1.12}$$

uniquely characterizes $(z - H)^{-1}$: if for some $z \notin \sigma(H)$ a bounded operator $R(z) : \mathcal{H} \to \mathcal{D}(H^0)$ satisfies (1.12), then [3] $R(z) = (z - H)^{-1}$.

For $N = 2$ all regularity properties of the resolvent kernel can be deduced from the Lippmann Schwinger equation

$$T(z) \equiv V + V\, R(z)\, V = V + V\, R_0(z)\, T(z)\,. \tag{1.13}$$

(1.13) can for $z \notin [0, \infty)$ be regarded as integral equation of second kind in $L^2(R^3)$:

$$T(k, q, z) = v\,(k - q) + \int dp\; v\,(k - p)\,(z - n\,p^2)^{-1}\, T(p, q, z) \tag{1.14}$$

where $v(k)$, the Fourier transform of $V(x)$, is in L^2 and $v\,(k - p)\,(z - n\,p^2)^{-1}$ a Hilbert Schmidt (HS) kernel. In particular, $\|\,V\,R_0(z)\,\|_{HS} \leq c\,(1 + |\,z\,|)^{-\delta}$, $\delta > 0$, for $\operatorname{Re} z \leq -1$. The Fredholm alternative applies. A solution $g \in L^2$ of the homogeneous equation $g = V\,R_0(z)\,g$ leads to an eigenstate $h = R_0(z)\,g$ of H with eigenvalue z. Therefore (1.14) has a unique solution $T(.\,, q, z) \in L^2$, if $z \notin \sigma(H)$. Actually $T(z)\,R_0(z) = (1 - V\,R_0(z))^{-1} V\,R_0(z)$ is as product of a bounded and a HS operator with $\|\,T(z)\,R_0(z)\,\|_{HS} \leq c\,(1 + |\,z\,|)^{-\delta}$ for $-\operatorname{Re} z$ sufficiently large.

For $N > 2$ the correct generalization of the equations (1.13) has been derived by FADDEEV [3] and YAKUBOVSKY [4]. Let α_k be a sequence of partitions

$$\alpha_k = (a_k, a_{k+1}, \ldots a_{N-1}), \quad \text{where} \quad a_m \supset a_{m+1}, \tag{1.15}$$

i.e. a_{m+1} is obtained from a_m by breaking up one of its groups. Let $a_i = (\mathfrak{a}_1, \ldots \mathfrak{a}_i)$ and

$$V(a_i) = \sum_{j=1}^{i} V_{\mathfrak{a}_j}, \quad H(a_i) = H^0 + V(a_i),\ R(a_i, z) = (z - H(a_i))^{-1}$$
$$V(a_i/a_{i+1}) = V(a_i) - V(a_{i+1})\,. \tag{1.16}$$

For any sequence α_k of partitions we define

$$T^{\alpha_k}(z) = V(a_{N-1})\,R(a_{N-1}, z)\,\ldots\,V(a_k/a_{k+1})\,(1 + R(z)\,V)$$
$$\tilde{T}^{\alpha_k}(z) = V(a_{N-1})\,R(a_{N-1}, z)\,\ldots\,V(a_k/a_{k+1})\,(1 + R(a_k, z)\,V(a_k))\,. \tag{1.17}$$

These operators have a simple graphical characterization. For sufficiently small $\operatorname{Re} z$, the Born series converges:

$$T(z) = V \sum_{n=0}^{\infty} (R_0(z)\,V)^n\,. \tag{1.18}$$

Any term $V_{ij}\,R_0\,V_{kl}\,\ldots\,R_0\,V_{mn}$ is in 1-1 correspondence with a graph $G = G(ij, \ldots mn)$, where N horizontal lines stand for particles $1, \ldots N$ and every V_{rs} is represented by a vertical connection of the lines r and s in the order $i\,j, \ldots m\,n$ from the left. $T^{\alpha_k}(z)$ is the sum of all graphs of $T(z)$ with at least connectivity α_k, i.e. $V_{ij} = V(a_{N-1})$, then the next potential not connecting the lines i and j belongs to $V(a_{N-2}/a_{N-1})$ etc. Similarly $\tilde{T}^{\alpha_k}(z)$ is the sum of all graphs in $V(a_k) + V(a_k)\,R(a_k, z)\,V(a_k)$ with connectivity α_k. By analytic continuation into the complement of $\sigma(H)$ one proves the "cluster decomposition"

$$T(z) = \sum_{\alpha_{N-1}} \tilde{T}^{\alpha_{N-1}}(z) + \ldots + \sum_{\alpha_3} \tilde{T}^{\alpha_3}(z) + \sum_{\alpha_2} T^{\alpha_2}(z)$$
$$T^{\alpha_2}(z) = \tilde{T}^{\alpha_2}(z) + T^{\alpha_1}(z)\,. \tag{1.19}$$

We decompose $T^{\alpha_1}(z)$:

$$T^{\alpha_1}(z) = \tilde{T}^{\alpha_2}(z)\,R_0(z)\,V(a_1/a_2)\,(1 + R(z)\,V) = \sum_{\beta_2} Q^{\alpha_2\beta_2}(z)\,R_0(z)\,T^{\beta_2}(z)\,. \tag{1.20}$$

Here the right-hand side is graphically defined as follows: Every graph G in $T^{\alpha_1}(z)$ can be cut into two graphs, $G = G_1\,G_2$, where the leftmost potential in G_2 is the first potential from the left in G belonging to $V(a_1/a_2)$. G_1 contributes to $\tilde{T}^{\alpha_2}(z)$ and is equal to $G_4\,G_3$, if one requires that G_3 has left connectivity β_3 and that the first potential to the right in G_4, $V(c_{N-1})$, connects b_3 to a_2, i.e. $c_{N-1} \not\subset b_3$, $c_{N-1} \subset a_2$. Then $G_3 G_2$ has the left connectivity β_2 with $b_2 \neq a_2$ and contributes to $T^{\beta_2}(z)$, and G_4 enters in $Q^{\alpha_2 \beta_2}(z)$.

For two sequences α_k, β_m define

$$\gamma_n = \alpha_k \llcorner \beta_m = (c_n, \dots c_{k-1}, \alpha_k) \tag{1.21}$$

as α_k completed by all new connections arising from β_m in the order $b_{N-1}, \dots b_m$. Then [5] $Q^{\alpha_2\beta_2}(z)$ is the sum of all graphs with left connectivity α_k, $2 \le k \le N-1$, and a rightmost potential $V(c_{N-1})$, $c_{N-1} \subset a_k$, such that $\alpha_k \llcorner \beta_3 = \alpha_2$, $c_{N-1} \not\subset b_3$, or in formulis:

$$Q^{\alpha_2\beta_2}(z) = \sum_{\substack{\alpha_k \llcorner \beta_3 = \alpha_2 \\ c_{N-1} \subset a_k,\, c_{N-1} \not\subset b_3}} \tilde{T}^{\alpha_k}(z)\, R_0(z)\, V(c_{N-1}) + \sum_{\substack{\alpha_k \llcorner \beta_3 = \alpha_2 \\ c_{N-1} \subset a_k,\, c_{N-1} \not\subset b_3, a_{k+1}}} \tilde{T}^{\alpha_{k+1}}(z)\, R_0(z)\, V(c_{N-1})\,. \tag{1.22}$$

In the sequel we shall use α instead of α_2. An algebraic derivation of the F-Y equations $(A^{\alpha\beta}(z) = Q^{\alpha\beta}(z)\, R_0(z))$

$$T^{\alpha}(z) = \tilde{T}^{\alpha}(z) + \sum_{\beta} A^{\alpha\beta}(z)\, T^{\beta}(z) \tag{1.23}$$

can be found in [4] together with the proof that the homogeneous equations

$$f^{\alpha} = \sum_{\beta} A^{\alpha\beta}(z)\, f^{\beta} \tag{1.24}$$

have only trivial solutions in L^2 for $z \notin \sigma(H)$. We shall use the F-Y equations to prove the

Theorem 1.2: Assume $V_{ij} \in L^2(R^3)$. Then H has below E^c only discrete eigenvalues E_{A_1} with finite multiplicities. The E_{A_1} are bounded from below and have at most E^c as accumulation point. For $z \notin [E^c, \infty)$, $R(z)$ is an integral operator which has for $z \notin [E, \infty)$, $E < E^c$, the representation

$$R(k, l, z) = \sum_{E_A < E} \frac{\psi_{A_1}(k)\, \psi_{A_1}(l)^*}{z - E_{A_1}} + \sum_{a} \delta_a\, (\bar{k} - \bar{l})\; R_a\, (\hat{k}, \hat{l}, z - E_a(\bar{k}))\,. \tag{1.25}$$

The kernels $R_a\, (\hat{k}, \hat{l}, z - E_a(\bar{k}))$ are holomorphic for $z \notin [E, \infty)$ and HS operators in the relative coordinates $\hat{k}, \hat{l}$ when multiplied with $(z - E(k))$ or $(z - E(l))$ with HS norm bounded by $c\,(1 + |\,z - E_a(\bar{k})\,|)^{-\delta}\ \delta > 0$, for Re z sufficiently small.

Remark: (1.9) and the first part of this theorem has been proved by Hunziker [6] under the much weaker assumption $V_{ij} \in L^2 + L^{\infty}$, if the L^{∞}-component of V_{ij} can be chosen arbitrarily small in the L^{∞}-norm.

Proof: Let us assume that Theorem 1.2 has already been proved for all subsystems $\mathfrak{a} \subsetneqq \{1, \dots N\}$. Then $A^{\alpha\beta}(z)$ is a bounded operator for $z \notin [E^c, \infty)$. For proving this, two remarks are helpful: Firstly, let $\mathfrak{a} = \{1, \dots r\}$ and $\hat{R}_{\mathfrak{a}}(\hat{k}_1, \dots \hat{l}_r, z)$ be the resolvent kernel of $\hat{H}_{\mathfrak{a}}$ in $\hat{\mathcal{H}}_{\mathfrak{a}}$. Then the kernel of $R_{\mathfrak{a}}$ in $\mathcal{H}_{\mathfrak{a}}$ becomes

$$R_{\mathfrak{a}}(k_1, \dots k_r, l_1, \dots l_r, z) = \delta\,(\bar{k}(\mathfrak{a}) - \bar{l}(\mathfrak{a}))\, \hat{R}_{\mathfrak{a}}(k_1 - \bar{k}(\mathfrak{a})\, m_1/m(\mathfrak{a}), \dots$$
$$\times\, \bar{l}_r - l(\mathfrak{a})\, m_r/m(\mathfrak{a}), z - n(\mathfrak{a})\, k(\mathfrak{a})^2)\,. \tag{1.26}$$

Secondly, let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be two disjoint subsystems. Then the resolvent of $H_1 + H_2$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be computed from the resolvents of H_i in $\mathcal{H}_i$ ($i = 1, 2$) by

$$R_{1+2}(z) = \frac{1}{2\pi i} \int_{\Gamma} dz\, R_1(z')\, R_2(z - z') \tag{1.27}$$

where Γ encircles the spectrum of H_1 in a counter clockwise way, sufficiently close to the real axis. (1.25) then leads to a bound of $A(z)$ as $0(|z|^{-\delta})$ for $\operatorname{Re} z \to -\infty$. According to (1.22) the $(N-1)^{th}$ power of the F-Y kernel is a finite sum of terms

$$\prod_{i=1}^{N-1} V_i\, R_0(z)\, W_i(z) \tag{1.28}$$

where the V_i belong to $V(a_{N-i}/a_{N-i+1})$ and the $W_i(z)$ are bounded operators of the form $\mathbf{1}$ or $\Pi\, V\, R(a, z)$ of at most connectivity a_{N-i}. One proves by induction [10] that for $z \notin [E^c, \infty)$ (1.28) is a HS operator with HS norm $0(|z|^{-\delta})$ as $\operatorname{Re} z \to -\infty$. By the Riesz-Schauder theorem, $(1 - A(z))^{-1}$ exists in $\mathfrak{L}(\mathcal{H})$, whenever there are no non-trivial solutions of the homogeneous F-Y equations. The latter are of finite multiplicity and in 1-1 correspondence with eigenstates of H. Since furthermore $(1 - A(z))^{-1}$ is meromorphic outside $[E^c, \infty)$ and holomorphic for $\operatorname{Re} z$ sufficiently small, the singular values E are real, bounded from below and have at most E^c as accumulation point. In a neighborhood of E, $R(k, l, z)$ has the representation

$$\sum_{E_{A_1}=E} \frac{\psi_{A_1}(k)\, \psi_{A_1}(l)^*}{z - E} + \tilde{R}(k, l, z) \tag{1.29}$$

where $\tilde{R}(z)$ is holomorphic in z. Elsewhere the square integrability of the solution follows from the L^2 properties of the inhomogeneity of the F-Y equation and the boundedness of $(1 - A(z))^{-1}$. Below any $E < E^c$, only a finite number of singular values occur, and the representation (1.25) is obtained using the Cauchy integral formula.

Theorem 1.3: For $V_{ij} \in L^2$ one has in the L^2 topology

$$\Omega_A^{\pm}\, f_A \otimes \psi_A(k) = \lim_{\varepsilon \downarrow 0} \mp i\, \varepsilon \int dl\, R\,(k, l, E_A(l) \mp i\, \varepsilon)\, f_A(l)\, \psi_A(l) \tag{1.30}$$

Proof: An immediate corrolary of Theorem 1.1 is the identity

$$\Omega_A^{\pm}\, f_A \otimes \psi_A = \underset{\varepsilon \downarrow 0}{s\text{-}\lim} \pm \varepsilon \int_0^{\pm\infty} dt \exp(i\, H\, t) \exp(-\, i(H_A^0 \mp i\, \varepsilon)t)\, f_A \otimes \psi_A\,. \tag{1.31}$$

We have to show that for $\varepsilon > 0$ (1.30) is related to the right-hand side of (1.31). Take any $g \in \mathcal{H}$ with compact spectrum of $H = \int \lambda\, dE(\lambda)$. Let Γ be a bounded smooth curve encircling the support of $dE(\lambda)\, g$ in $\{|\operatorname{Im} z| < \varepsilon/2\}$. Then

$$\begin{aligned} &\int_{-\infty}^{0} dt\, e^{\varepsilon t}\, (g, \exp(i\, H\, t) \exp(-\, i\, H_A^0\, t)\, f_A \otimes \psi_A) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{0} dt\, e^{\varepsilon t} \int_{\Gamma} dz\, e^{itz} \int dk\, dl \exp(-\, i\, E_A(l) t) \\ &\times g(k)^*\, R(k, l, z)\, f_A(l)\, \psi_A(l)\,. \end{aligned} \tag{1.32}$$

By theorem 1.2, all integrations in (1.32) are absolutely convergent and by Fubini's theorem independent of the order of integration.

So far the elementary connection between scattering theory and the F-Y equations.

§ 2. Asymptotic Completeness and the Singular Limit of the F-Y Equations

In nonrelativistic transcription the axioms of S-matrix theory [13] would be

(1) Connectedness (vacuum cluster properties)
(2) Galilei invariance
(3) Unitarity
(4) Maximal regularity.

The vacuum cluster properties have been deduced by HUNZIKER [11]. The scattering amplitudes are Galilei invariant, if the V_{ij} are rotation invariant (see e.g. [12]). Maximal regularity is a well-developed concept only as far as the regularity properties of the scattering amplitudes in the physical region are concerned. Here certain singularities (solutions of the Landau equations for positive α-parameters) are necessary for a macroscopic space-time description of scattering [9].

In the following we shall study only N-body systems with a finite number of channels, all strictly below the continuum. We shall require (see section 5):

(S): For all subsystems $\mathfrak{a} \subseteq \{1, \ldots N\}$, $H_{\mathfrak{a}}$ has no discrete spectrum for $E \geq E^c_{\mathfrak{a}} - \delta$, $\delta = \delta(\mathfrak{a}) > 0$.

Consider generalized Feynman integrals with smooth numerator functions and any sequence of energy denominators $(z - E_A(q))^{-1}$ for arbitrary channels A (consistent with the occurrence in an iteration of the F-Y equations). The physical region Landau singularities [9] are exactly those of all generalized Feynman integrals. Maximal regularity is the postulate that the physical scattering amplitudes or, more generally, the resolvent kernels are essentially as regular as in perturbation theory and of similar asymptotic behaviour at infinity. We shall outline how a weak form of maximal regularity implies unitarity.

Theorem 1.3 contains some information about the singularity structure of $R(k, l, z)$ for $\operatorname{Im} z \to 0$. Because of $R(k, l, z) = R(l, k, z^*)^*$, there must be multiplicative singularities

$$R(k, l, z) = \sum_A \frac{\psi_A(k)\, \psi_A(l)^*\, \delta_A\,(k-l)}{z - E_A(k)} + \sum_{a(A),\, a(B) \neq a_1} \frac{\psi_A(k)\; T_{AB}(k, l, z)\; \psi_B(l)^*}{(z - E_A(k))\;(z - E_B(l))}. \tag{2.1}$$

For $f \in \mathcal{D}(R_0^{3N})$ and any A, $a(A) \neq a_1$, we define

$$\Phi_A(f)\;(z, k) = \psi_A(k) \int dl\; f(l) \left\{ \psi_A(l)^*\, \delta_A(k - l) + \sum_{a(B) \neq a_1} \frac{T_{AB}\,(k, l, z)\; \psi_B(l)^*}{(z - E_B(l))} \right\}. \tag{2.2}$$

The following assumption can be easily proved for every term in generalized perturbation theory, if the numerator functions are Hölder continuous (HC) and sufficiently decreasing at infinity:

(R): $\Phi_A(f)\;(z, .) \in B(\theta, \mu)$ for some $\theta > 3/2$, $\mu, \nu > 0$, uniformly for $\operatorname{Re} z$ bounded from above, $\operatorname{Im} z \geq 0$ or ≤ 0, $|w| \leq 1$, $\operatorname{Im} z\; \operatorname{Im} w \geq 0$:

$$\| \Phi_A(f)\;(z, .) \|_{\theta,\mu} \leq C \qquad \| \Phi_A(f)\;(z + w, .) - \Phi_A(f)\;(z, .) \|_{\theta,\mu} \leq C\, |w|^{\nu} \tag{2.3}$$

(the Banach space $B(\theta, \mu)$ is defined in App. A).

Theorem 2.1: Assume that $R(k, l, z)$ has the form (2.1) with $\Phi_A(f)\,(z, k)$ satisfying (R). Then asymptotic completeness holds:

$$\mathbf{1} = \sum_A \Omega_A^{ex}\,(\Omega_A^{ex})^*\,, \qquad \mathrm{ex} = \pm\,. \tag{2.4}$$

Proof: By Theorem 1.1 $\Omega_A^{ex}(\Omega_A^{ex})^*$ is the projector on $\mathcal{H}_A^{ex}$. We shall relate (2.4) to the resolution of the identity of H. For $f, g \in \mathcal{D}$, one has in the weak topology

$$\begin{aligned}\big(f, dE(\lambda)\, g\big) &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\,\pi\,i}\,\big(f, [R(\lambda - i\,\varepsilon) - R\,(\lambda + i\,\varepsilon)]\, g\big)\, d\lambda \\ &= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\,\big(f, R\,(\lambda \mp i\,\varepsilon)\, R\,(\lambda \pm i\,\varepsilon)\, g\big)\, d\lambda\,.\end{aligned} \tag{2.5}$$

We insert for $h = f, g$

$$R\,(\lambda \pm i\,\varepsilon) h(k) = \sum_{a(A) \neq a_1} \frac{\phi_A(h)\;(\lambda \pm i\,\varepsilon, k)}{(\lambda \pm i\,\varepsilon - E_A(k))} + \sum_{A_1} \frac{(\psi_{A_1}, h)\;\psi_{A_1}(k)}{(\lambda \pm i\,\varepsilon - E_{A_1})}\,. \tag{2.6}$$

The diagonal terms contribute to (2.5) for $\varepsilon \downarrow 0$

$$\begin{aligned}&\sum_{a(A) \neq a_1} \int dk\,\delta\,(\lambda - E_A(k))\;\Phi_A(f)\,\big(E_A(k) \mp i\,0, k\big)^*\,\Phi_A(g)\,\big(E_A(k) \mp i\,0, k\big)\, d\lambda \\ &\qquad + \sum_{A_1} (\psi_{A_1}, f)^*\,(\psi_{A_1}, g)\;\delta\,(\lambda - \mathrm{E}_{A_1})\, d\lambda \\ &\qquad = \sum_{a(A) \neq a_1} \int dk\,\delta\,(\lambda - E_A(k))\,(\Omega_A^{ex\,*} f)\,(k)^*\,(\Omega_A^{ex\,*} g)\,(k)\, d\lambda \\ &\qquad + \sum_{A_1} (\psi_{A_1}, f)^*\,(\psi_{A_1}, g)\;\delta\,(\lambda - E_{A_1})\, d\lambda\,.\end{aligned} \tag{2.7}$$

In the last step we observe that after inserting (2.1) and (2.2) into (1.30) one obtains

$$\begin{aligned}(\Omega_B^{\pm\,*}\, h, f_B \otimes \psi_B) &= \lim_{\varepsilon\downarrow 0} \mp\, i\,\varepsilon \int dk\,\big[\int dl\; R(k, l, E_B(k) \pm i\,\varepsilon)\; h(l)\big]^*\, f_B(k)\;\psi_B(k) \\ &= \int dk\;\Phi_B(h)\;(E_B(k) \pm i\,0, k)^*\, f_B(k)\;\psi_B(k)\,.\end{aligned} \tag{2.8}$$

Here the terms in the representation (2.6) with $A \neq B$ do not contribute: If $a(A) = a(B)$, then ψ_A and ψ_B are orthogonal. Otherwise $E_A(k) - E_B(k)$ is non-degenerate and the numerator HC and therefore the integral $o(1)$ for $\varepsilon \downarrow 0$. In particular we see that $\Phi_B(h)\;(E_B(k) \pm i\,0, k)$ is square integrable in k. For the same reason as in (2.8), the non-diagonal terms do not contribute in (2.5). After integration over λ we obtain (2.4) between a dense set of states in $\mathcal{H}$.

For the proof of maximal regularity the amplitudes in the F-Y equations have to be more differentiated. With the help of the spectrum condition (S) one can separate the multiplicative singularities (2.1) in the kernels of T^α, $\tilde{T}^\alpha$ (1.17) and of $A^{\alpha\beta}$ (1.22):

$$\begin{aligned}T^\alpha(k, l, z) &= T^\alpha_{A_N D_N}(k, l, z) + \Sigma'\, T^\alpha_{A D_N}(k, l, u\, z)/\big(z - E_A(k)\big) \\ &\quad + \Sigma'\, T^\alpha_{A_N D}(k, l, z)/\big(z - E_D(l)\big) + \Sigma'\, T_{AD}(k, l, z)/\big(z - E_A(k)\big)\;(z - E_D(l))\end{aligned} \tag{2.9}$$

Σ' denotes summation over all different channel energies E_A, $E_D \neq E_{A_1}$, E_{A_N}. The summation over A is further restricted by requiring consistency with α, i.e. $a(A) = a_i$ for some i. $\tilde{T}^\alpha(k, l, z)$ has a similar structure, and

$$\begin{aligned}A^{\alpha\beta}(k, l, z) &= \Sigma'\, Q^{\alpha\beta}_{A_N C}(k, l, z)/\big(z - E_C(l)\big) \\ &\quad + \Sigma'\, Q^{\alpha\beta}_{AC}(k, l, z)/\big(z - E_A(k)\big)\;\big(z - E_C(l)\big)\,.\end{aligned} \tag{2.10}$$

The A are restricted as above, while the C have to satisfy $a(C) \neq a_1$, $a(C) \subseteq a_2$ with a $c_{N-1} \subseteq a(C)$ such that $c_{N-1} \llcorner\, \beta_2 = \beta_1$, if $a(C) \neq a_N$. Written in these components the F-Y equations become

$$T^{\alpha}_{AD}(k, l, z) = \tilde{T}^{\alpha}_{AD}(k, l, z) + \sum_{\beta, C} \int dp\; Q^{\alpha\beta}_{AC}(k, p, z)\; T^{\beta}_{B_N D}(p, l, z)/(z - E_C(p))$$

$$+ \sum_{\beta, C, B} \int dp\; Q^{\alpha\beta}_{AC}(k, p, z)\; T^{\beta}_{BD}(p, l, z)/(z - E_C(p))\,(z - E_B(p))\,. \tag{2.11}$$

The singular denominators indicate to study (2.11) in the Banach space $B(\theta, \mu)$ (see App. A). For HC integrands $F(k, p, z)$ of sufficient decrease at infinity, the double singular integral

$$G(k, z) = \int dp\; F(k, p, z)/(z - E_C(p))\,(z - E_B(p)) \tag{2.12}$$

is again HC in k and z for $\operatorname{Im} z \geq 0$ or ≤ 0 (Either $a(C) \neq a_N$, and, after a linear change of variables, the denominators become $(z - q_1^2 - E_1(q_2, \ldots q_{N-1}))\,(z - q_2^2 - E_2(q_3, \ldots q_{N-1}))$, where E_1, E_2 are polynomials. Privalov's lemma [3] can then be applied first to the q_1- and then to the q_2-integration. Or $a(C) = a_N$, and then $E_C(p) - E_B(p) > E$ for some $E > 0$).

The Faddeev program starts from 2-body potentials $v_{ij} \in B(\theta, \mu)$, $\theta > 3/2$, $\mu > 1/2$. The hope is to solve (2.11) in some $B(\alpha, \beta)$, $\alpha > 3/2$, $\beta > 0$, uniformly for $\operatorname{Im} z \to 0$. There are, however, several serious difficulties:

(1) The inhomogeneity $\tilde{T}$ does not belong to any $B(\alpha, \beta)$. Perturbation theory suggests that ${}^K\tilde{T}$, the inhomogeneity of the K^{th} iteration of (2.11), $K \geq K_0$, belongs to some $B(\alpha, \beta)$, if the spectrum condition (S) is satisfied.

(2) The kernel $A(z)$ of (2.11) is not a bounded operator in any $B(\alpha, \beta)$, for $\alpha > 3/2$, $\beta > 0$ and $\operatorname{Im} z \to 0$. This is due to the fact that the amplitudes $Q_{AC}(z)$ have additional singularities (δ-functions for disconnected components and rescattering singularities). From perturbation theory we expect:

(A) For some $\gamma > \alpha > 3/2$, $\delta > \beta > 0$, all powers $A^K(z)$ exist as bounded linear operators $B(\gamma, \delta) \to B(\alpha, \beta)$, uniformly for $\operatorname{Im} z \geq 0$ or ≤ 0, if $\operatorname{Re} z$ is bounded from above.

(B) For some K_0 and all $K \geq K_0$, $A^K(z)$ can be extended to a bounded linear operator: $B(\alpha, \beta) \to B(\gamma, \delta)$, uniformly for $\operatorname{Im} z \geq 0$ or ≤ 0, if $\operatorname{Re} z$ is bounded from above.

Property (A) will be a consequence of Privalov's lemma, once the kernels $Q_{AC}(z)$ can be exhibited as a finite sum of generalized Feynman integrals with HC numerators of uniform asymptotic decrease. (B) follows by (2.12), if for $K \geq K_0$ the kernels ${}^KQ_{AC}(z)$ are HC and of uniform decrease at infinity. Then $A^K(z)$ is compact and the Fredholm alternative holds for (2.11), although $A(z)$ has not even a dense domain in $B(\alpha, \beta)$ [14]:

Theorem 2.2: Under the assumptions (A) and (B), either $f = Af$ has a non-trivial solution in $B(\gamma, \delta)$, or $f = g + Af$ has a unique solution in $B(\alpha, \beta)$ for every $g \in B(\gamma, \delta)$.

(3) One has to control the Fredholm alternative in (2.11). Due to the inversion formulae in [4], solutions of the homogeneous equations can occur only for $z < E^c$

or for $z = E \pm i\,0$, $E \in [E^c, \infty)$. FADDEEV has shown for $N = 3$ and $\mu > 1/2$ that there are only countably many singular values of z with possible accumulation points only at the E_{A_2} and that except for these values all solutions of the homogeneous F-Y equations lead to eigenvectors of H. Thus the control over the Fredholm alternative can be related to the knowledge of the discrete spectrum of H, but not yet in a satisfactory way (see section 5). We shall always make the assumption

$(\bar{S}_N)$ If for some $\gamma > \alpha > 3/2$, $\delta > \beta > 0$ (A) and (B) are satisfied, then there are no non-trivial solutions $f = A(z)\,f$ in $B(\gamma, \delta)$ for $z = E \pm i\,0$, $E \in [E^c, \infty)$.

As outlined before, $(\bar{S})_N$ implies (S_N). In the following section a study of the 4-body problem with finitely many channels will be given in this spirit.

§ 3. Time-Dependent Approach to the Singular F-Y Equations

In this section we shall investigate the singular F-Y equations in the scale $B(\theta, \mu)$ of Banach spaces using a time-dependent method for proving the necessary regularity properties of the kernels Q and the inhomogeneities T. For weakly interacting 2-particle systems, PROSSER [7] and KATO [8] have derived asymptotic completeness from the convergence of the Dyson series for Ω^{ex} in certain L^p topologies. Essential is here the decay of free wave packets $\exp(-\,i\,H^0\,t)\,f$ for large t. We shall use similar techniques to prove that rescattering processes with highly connected classical orbits [9] cannot lead to long-range correlations in x-space or to strong singularities in p-space.

Consider for instance a rescattering contribution to the 3-body $T(z)$ for $V_{ij} \in S$ and $\operatorname{Im} z > 0$:

$$V_{12}\,R_0(z)\,V_{13}\,R_0(z)\,V_{23}\,R_0(z)\,V_{12}\,R_0(z)\,V_{23}$$

$$= \iint_0^\infty \prod_{i=1}^{4} dt_i \exp(i\,z\,\Sigma\,t_i)\;V_{12}\,e^{-it_1 H^0}\,V_{13}\,\ldots\,e^{-it_4 H^0}\,V_{23}\,. \tag{3.1}$$

For $\operatorname{Im} z > 0$ the t-integrals converge in the uniform topology. We are interested in the behaviour of the kernel of (3.1) for $\operatorname{Im} z \downarrow 0$. We obtain from Figure 1

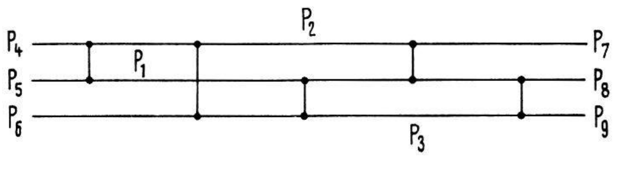

Figure 1

$$\iint_0^\infty d^4t \exp\,(i\,z\,\Sigma\,t_k) \int \prod_{j=1}^{3} dp_j\;F(p) \exp(-\,i\,\Sigma\,A_{ij}(t)\,p_i\,p_j)\,, \tag{3.2}$$

where $F(p)$ arises from the potentials

$$\begin{aligned} F(p) = {} & v_{12}\,(p_1 - p_5)\,v_{13}\,(-\,p_1 - p_2 - p_6) \\ & \times v_{23}\,(p_1 + p_2 + p_3)\,v_{12}\,(p_2 - p_7)\,v_{23}\,(p_9 - p_3) \end{aligned} \tag{3.3}$$

and the quadratic form $A_{ij}(t)$ comes from the energies of the intermediate states.

For $m_i = 1/2$, $E(p, q) = p^2 + q^2 + (p + q)^2$ and

$$\Sigma A_{ij}(t)\, p_i\, p_j = t_1\, E(p_1, p_6) + t_2\, E(p_1, p_2) + t_3\, E(p_2, p_3) + t_4\, E(p_3, p_7)\,. \tag{3.4}$$

We expect continuity in the external momenta $p_4, \ldots p_9$ and in z for $\operatorname{Im} z \downarrow 0$, if in x-space the multiple rescattering (3.1) becomes sufficiently improbable for large separations. There should be a $\Pi\,(1 + \lambda_k)^{-3/2}$ decrease in the scaling parameters $\lambda_1, \lambda_2, \lambda_3$ of the three independent closed loops in Figure 1. The absolute convergence of the t-integration for $\operatorname{Im} z \downarrow 0$ depends critically on λ_k as functions of $t_1 \ldots t_4$. For (3.2) we expect bounds $(1 + t_1 + t_2)^{-3/2}\,(1 + t_2 + t_3)^{-3/2}\,(1 + t_3 + t_4)^{-3/2}$, which would be sufficient. Obviously convergent diagrams have to be highly connected, and such graphs are generated by the higher powers of the F-Y kernels.

Let $B(\theta, r)$, $\theta > 0$, $r = 1, 2, \ldots$ be the Banach space of all C^r functions $f : R^3 \to C$ with

$$\sup\,(1 + |\,p\,|)^\theta \sum_{|\varrho| \leq r} |\,D^\varrho f(p)\,| < \infty \tag{3.5}$$

(D^ϱ: differential monomial of degree ϱ_i in $\partial/\partial p_i$, $\Sigma \varrho_i = |\,\varrho\,|$) and $B(\theta, \infty) = \bigcap_r B(\theta, r)$.

Theorem 3.1: Let $v_{ij} \in B(\theta, \infty)$, $1 \leq i < j \leq N$, with $\theta > 3/2$. Then there exist $\alpha = \alpha(N) > 3/2$, $\beta = \beta(N) > 0$, $L = L(N) < \infty$, such that the kernel $G(k, l, z)$ of any graph G in (1.18) contributing to $A^{M-1}\,Q$, $M \geq L$, has a limit for $\operatorname{Im} z \to 0$ and satisfies

$$\sup N(k, \alpha)^{-1}\, N(l, \alpha)^{-1}\,(1 + E(l))^{-1} \left\{ |\,G(k, l, z)\,| + \frac{|\,G\,(k+h,\, l+m,\, z+w) - G\,(k, l, z)\,|}{|\,h\,|^\beta + |\,m\,|^\beta + |\,w\,|^\beta} \right\} < \infty \tag{3.6}$$

Here the supremum extends over all $h, k, l, m \in R_0^{3N}$ with $|\,h\,|, |\,m\,| \leq 1$ and over all $z, w \notin [0, \infty)$ with $|\,w\,| \leq 1$, $\operatorname{Im} z \operatorname{Im} w \geq 0$ and $\operatorname{Re} z \leq E_0$, E_0 arbitrary, but fixed.

Proof: Consider any connected graph of order K in (1.18) for $\operatorname{Im} z > 0$. It is convenient to separate in every resolvent kernel $(z - E(q))^{-1}$ the dominant contribution for large momenta from the local singularities for $\operatorname{Im} z \downarrow 0$ by a C^∞ partition

$$\begin{aligned} &1 = \chi_1(z) + \chi_2(z)\,, \qquad z \in C \\ &\operatorname{supp} \chi_1 \subset \{\,|\,z\,| < 2\,\delta\} \\ &\operatorname{supp} \chi_2 \cap \{\,|\,z\,| \leq \delta\} = \varnothing \end{aligned} \tag{3.7}$$

where $\delta > 0$ is arbitrary but fixed. We introduce $\varrho_i(z) = z^{-1} \chi_i(z)$, $i = 1{,}2$. $\varrho_2\,(z - E(q))$ is C^∞ and decreases uniformly for $\operatorname{Re} z \leq E_0$ with all derivates as $(1 + E(q))^{-1}$, while $\varrho_1\,(z - E(q))$ has compact support in q and will be represented as a t-integral (3.1). Expand the kernel of G into a sum of $\varrho_1\,\varrho_2$-monomials. After an allowed interchange of the t- and loop-integrations we obtain typically

$$\int d^m t \exp\left(i\, z \sum_{\mu=1}^{m} t_\mu\right) G(p_{L+1}, \ldots p_{L+2N}, t_1, \ldots t_m, z) \tag{3.8}$$

$$G(p, t, z) = \int \prod_{m=1}^{L} dp_m\, F(p_1, \ldots p_{L+2N}, z) \exp\left(-\, i \sum_{m,n=1}^{L+2N} A_{mn}(t)\, p_m\, p_n\right). \tag{3.9}$$

Here $p_1, \ldots p_L$ are loop momenta and $(p_{L+1}, \ldots p_{L+N}), (p_{L+N+1}, \ldots p_{L+2N}) \in R_0^{3N}$ external momenta. $F(p, z)$ incorporates the products of potentials, ϱ_2-terms and χ_1-factors, and $A_{mn}(t)$ is linear in t and depends on the connectivity of the graph.

It will become clear from the following estimates that the worst local singularities arise from the pure ϱ_1-term, while the relevant asymptotic behaviour for large external momenta is determined by the pure ϱ_2-term. The discussion of these two cases requires different techniques.

Pure ϱ_1-terms: In K^{th} order there are K t-integrations and $L = K + 2 - N$ loop integrations. We require that G contributes to $A^{M-1}Q$, where $M \geq 3N - 4$. The loop momenta $p_1, \ldots p_L$ will be placed consistently on certain internal particle lines: p_λ on line $h(\lambda)$ between the $i(\lambda)^{th}$ and $f(\lambda)^{th}$ interaction. These lines contribute to $(p, A\,p) = \Sigma A_{mn}\, p_m\, p_n$ a diagonal form

$$(p, B\,p) = \sum_{\lambda=1}^{L} B_\lambda\, p_\lambda^2, \qquad B_\lambda = n_{h(\lambda)} \sum_{k=i(\lambda)}^{f(\lambda)-1} t_k \tag{3.10}$$

The remaining internal lines give a positive semidefinite form $(p, C\,p) = (p, A\,p) - (p, B\,p)$. We maximize the B_λ by choosing different sets of loop momenta in different sectors

$$t_{\varrho(1)} \geq t_{\varrho(2)} \geq \ldots \geq t_{\varrho(K)} \tag{3.11}$$

of the t-integration. We place the maximal number $\nu(1)$ of loop momenta on lines through intermediate state $\varrho(1)$, then $\nu(2) - \nu(1)$ through $\varrho(2)$, etc. Then A_{mn} will be partially diagonalized:

$$(p, A\,p) = A_{11}\left(p_1 + A_{11}^{-1} \sum_{j=2}^{L+2N} A_{1j}\, p_j\right)^2 + \sum_{i,j=2}^{L+2N} A_{ij}\, p_i\, p_j - A_{11}^{-1}\left(\sum_{j=2}^{L+2N} A_{1j}\, p_j\right)^2 . \tag{3.12}$$

If we set

$$\begin{aligned} \overline{p}_1 &= p_1 + A_{11}^{-1} \sum_{j=2}^{L+2N} A_{1j}\, p_j \\ A'_{ji} &= A_{ij} - A_{1j}\, A_{1i}\, A_{11}^{-1} \qquad 2 \leq i, j \leq L + 2N \end{aligned} \tag{3.13}$$

then (3.12) becomes $A_{11}\, \overline{p}_1^2 + \Sigma A'_{ij}\, p_i\, p_j$, and by the Schwarz inequality $C'_{ij} = A'_{ij} - \delta_{ij}\, B_i$ is again positive semidefinite. Recursively we obtain

$$\begin{aligned} \overline{p}_k &= p_k + \sum_{j=k+1}^{L+2N} e_{kj}\, p_j , \qquad 1 \leq k \leq L \\ \overline{p}_k &= p_k , \qquad L + 1 \leq k \leq L + 2N \\ (p, Ap) &= \sum_{k=1}^{L} D_k\, \overline{p}_k^2 + \sum_{i,j=L+1}^{L+2N} \overline{A}_{ij}\, \overline{p}_i\, \overline{p}_j \end{aligned} \tag{3.14}$$

where for all t and all $1 \leq k \leq L : D_k \geq B_k$. The shifts e_{kj} can be uniformly estimated in the sector (3.11): For the first $\nu(1)$ internal momenta one has $B_j \geq t_{\varrho(1)} \min\{n_i\}$, $1 \leq j \leq \nu(1)$, and

$$A_{jj} \geq B_j \geq c \max_{1 \leqslant i \leqslant L+2N} |A_{ij}| \tag{3.15}$$

for some $c > 0$ independent of t. Let $g > 1$ be the smallest integer, such that $\nu(g) - \nu(g-1) > 0$. The momenta of the intermediate states $\varrho(1), \ldots \varrho(g-1)$ are by construction independent of $p_{\nu(g-1)+1}, \ldots p_{\nu(g)}$. Therefore the A_{ij} with i or $j \geq \nu(g-1) + 1$ are linear combinations of the $t_{\varrho(g)}, \ldots t_{\varrho(K)}$. By induction one proves that (3.15) holds for all $1 \leq j \leq L$ with $c > 0$ independent of t. Furthermore a similar relation holds when going over from A to A', A'', ..., due to the uniform boundedness of $A_{11}^{-1} A_{1m}$, $(A'_{22})^{-1} A'_{2n}$, ... Therefore $|e_{kj}| \leq d < \infty$ uniformly for all t in (3.11). In the new coordinates, (3.9) becomes

$$\int \prod_{m=1}^{L} dp_m F\left(\sum_{m=1}^{L+2N} f_{1m} \bar{p}_m, \ldots \sum_{m=L}^{L+2N} f_{Lm} \bar{p}_m, \bar{p}_{L+1}, \ldots \bar{p}_{L+2N}, z\right)$$
$$\times \exp\left(-i \sum_{m=1}^{L} D_m \bar{p}_m^2 - i \sum_{i,j=L+1}^{L+2N} \bar{A}_{ij} \bar{p}_i \bar{p}_j\right) \tag{3.16}$$

where the f_{ij} in the inversion of (3.14) are again uniformly bounded in t. In (3.16), we split off the factor $\exp(-i(\bar{p}, \bar{A}\bar{p}))$. $F(p, z)$ has for $\operatorname{Re} z \leq E_0$, due to the χ_1-factors, compact support in all $\bar{p}_1, \ldots \bar{p}_{L+2N}$, except for the relative momenta appearing in the first and last potentials, where (3.5) holds. Thus after applying to the remainder in (3.16) any differential monomial D with respect to z and $\bar{p}_{L+1}, \ldots \bar{p}_{L+2N}$ we obtain by standard estimates [15]

$$\left|\int d^{3L}p \, D \, F(\bar{p}, t, z) \exp\left(-i \sum_{k=1}^{L} D_k \bar{p}_k^2\right)\right|$$
$$\leq c \prod_{k=1}^{L} (1 + D_k)^{-3/2} N(\bar{p}_{L+1}, \ldots \bar{p}_{L+N}, \theta) \, N(\bar{p}_{L+N+1}, \ldots \bar{p}_{L+2N}, \theta) \,. \tag{3.17}$$

We now use the following lemma (to be proved in App. 3).

Lemma 3.2: There exists a $\delta > 0$, such that every N-particle graph contributing to $A^{M-1}Q$, $M \geq 3N - 4$, satisfies in every sector (3.11)

$$\prod_{k=1}^{L} (1 + D_k)^{-3/2} \leq \prod_{k=1}^{K} (1 + t_k)^{-1-\delta} \,. \tag{3.18}$$

We have to obtain uniform estimates for the Hölder differences of (3.8) in $p_{L+1}, \ldots p_{L+2N} \in R_0^{3N} \times R_0^{3N}$ and z, $\operatorname{Re} z \leq E_0$, $\operatorname{Im} z \geq 0$. Using the t-integral representation and (3.18), all non-trivial majorizations can be reduced to

$$\iint_{(3.11)} d^K t \prod_{k=1}^{K} (1 + t_k)^{-1-\delta} \left| \exp(i \Sigma t_k (z - w)) - \exp(i \Sigma t_k z) \right| \tag{3.19}$$

$$\iint_{(3.11)} d^K t \prod_{k=1}^{K} (1 + t_k)^{-1-\delta} \left| \exp(i \bar{A}_{ij} p_i h_j) - 1 \right|$$

$$\iint_{(3.11)} d^K t \prod_{k=1}^{K} (1 + t_k)^{-1-\delta} \left| \exp(i \bar{A}_{ij} h_i h_j) - 1 \right| . \tag{3.20}$$

For (3.19) we use the mean value theorem, when $|\Sigma t_k| \leq |w|^{-1}$: $|\exp(i \Sigma t_k (z+w)) - \exp(i \Sigma t_k z)| \leq |\Sigma t_k w|$ and obtain for $0 < \beta < \delta$

$$|w|^{-\beta} \left| \int\limits_{\Sigma t \leqslant |w|^{-1}} (3.19) \right| \leq \iint\limits_0^\infty d^K t \, (\Sigma t)^\beta \prod_{k=1}^K (1+t_k)^{-1-\delta} . \tag{3.21}$$

In $|\Sigma t| \geq |w|^{-1}$, we bound the exponentials by 1 and get the same extimate. Similarly one obtains for (3.20)

$$|(3.20)| \leq c \, |h_j|^\beta (|h_i|^\beta + |p_i|^\beta) . \tag{3.22}$$

Therefore (3.6) holds, whenever $0 < \beta < \delta$ is so small that $\alpha = \theta - \beta > 3/2$.

Pure ϱ_2-term: Here Weinberg's theorem [16] is applicable. However, it is difficult to define for an arbitrarily complicated graph a simple algorithm. We shall therefore prove in App. 3 the very elementary and by no means optimal

Lemma 3.3: There exists a $\alpha > 3/2$, such that every pure ϱ_2-term of any graph G contributing to $A(z)^M$, $M \geq 4\,(N-1)$, is uniformly bounded in the external momenta $k, l \in R_0^{3N}$ by $c\,N(k, \alpha)$ $(N(l, \alpha)$, where c depends on G and on max $\{0, \operatorname{Re} z\}$.

Mixed expressions are locally better behaved than the ϱ_1-term and at infinity better that the ϱ_2-term. This observation concludes the proof of Theorem 3.1.

It is difficult to extend this perturbation-theoretic argument to the exact kernel of the N-body F-Y equations. Firstly, additional singularities arise from bound state poles for all channels A_i with $2 \leq i \leq N-1$. If the spectrum condition (S) is satisfied, then Theorem 3.1 can be extended to any generalized Feynman amplitude with numerators from $B(\theta, \infty)$ and any consistent inclusion of bound state poles. The second difficulty is more serious: even after having removed a finite number of iterations of the F-Y equations there remain threshold singularities in subenergies

$$z = \sum_{j=1}^{i} n(\mathfrak{a}_j)\, \bar{k}(\mathfrak{a}_j)^2 + E_A \qquad z = \sum_{j=1}^{i} n(\mathfrak{a}_j)\, \bar{l}(\mathfrak{a}_j)^2 + E_A \tag{3.23}$$

where the remaining kernels cannot be C^∞. Here $a_i = \{\mathfrak{a}_1, \ldots \mathfrak{a}_i\}$ is any partition and E_A the energy of a channel A with $a_i \supseteq a(A)$. For $a_i = a(A)$, the singularity is multiplicative (see (2.1)).

There are two alternatives. Either one is strong enough to discuss the real singularities of Feynman integrals with HC numerators of compact support (as Faddeev for $N = 3$). Or one has to exhibit more explicitly the special structure of the threshold singularities in the numerators. By a gracious act of Fortuna this second method is fairly simple for $N = 3$ and 4 and smooth potentials. Here one has only to deal with the threshold singularity of the 2-body scattering amplitude $T_{ij}(k, l, z)$ at $z = 0$. For simplicity, we formulate our results for $v_{ij} \in B(\theta, \infty)$, leaving as an exercise the generalization to $v_{ij} \in B(\theta, r)$, $r = r(N)$ sufficiently large.

We start by accumulating information about the 2- and 3-body problem, supplementing the results of Povzner [17], Ikebe [18] and Faddeev [3] with emphasis on maximal regularity.

Theorem 3.4: Assume $v \in B(\theta, \infty)$, $\theta > 3/2$, and $(\bar{S}_2)$. Let $\{\psi_n\}$ be an eigenbasis of H, $H \psi_n = E_n \psi_n$. Then the 2-body off-shell scattering amplitude $T(k, l, z)$ in the relative coordinates has the form

$$T(k, l, z) = \sum_n \frac{\psi_n(k) \psi_n(l)^*}{z - E_n} + \hat{T}(k, l, z) \tag{3.24}$$

where for some $\alpha > 3/2$

(a) $\varphi_r(k) = \psi_r(k) (E_r - n k^2) \in B(\alpha, \infty)$

(b) $\hat{T}(k, l, z)$ is C^∞ in $k, l \in R^3$ and holomorphic for $z \notin [0, \infty)$. $\hat{T}(k, l, z)$ and all its derivatives with respect to k, l and Re z are uniformly bounded by $c (1 + | k - l |^{-\alpha})$ for Re z bounded from above, Im $z \geq 0$ or ≤ 0 and z outside of a neighborhood of $z = 0$.

(c) In any neighborhood N of $z = 0$ for any integer $r \geq 0$, there is a splitting $\hat{T} = \hat{T}_1 + \hat{T}_2$. $\hat{T}_1$ is a finite sum of terms

$$\int\!\!\cdots\!\!\int_0^\infty d^m t \exp\left(i z \sum_{r=1}^m t_r\right) T_s(k, l, t_1, \ldots t_m, z) \tag{3.25}$$

with supp $T_s \subset R^6 \times [0, \infty)^m \times N$, $m = m(s)$, T_s is C^∞ in k, l, z and with all derivatives bounded by $c (1 + | k - l |)^{-\alpha} \Pi (1 + t_r)^{-3/2}$. $\hat{T}_2$ is C^r in k, l and Re z for Im $z \geq 0$ or ≤ 0 the with same bounds as in (b).

Proof: The existence of a solution $T(k, l, z)$ of the form (3.24) has been established by FADDEEV within the class of HC functions of decrease $\alpha > 3/2$. The equations

$$\begin{aligned} \varphi_m(k) &= \int dl\, v (k - l) (E_m - n l^2)^{-1} \varphi_m(l) \\ T(k, l, z) &= v (k - l) + \int dp\, v (k - p) (z - n p^2)^{-1} v (p - l) \\ &\quad + \iint dp\, dq\, v (k - p) (z - n p^2)^{-1} T(p, q, z) (z - n q^2)^{-1} v (q - l) \end{aligned} \tag{3.26}$$

show differentiability in k, l and analyticity in $z \notin [0, \infty)$.

$$\frac{\partial}{\partial z} T(k, l, z) = - \int dp\, T(k, p, z) (z - n p^2)^{-2} T(p, l, z) \tag{3.27}$$

and iterations prove differentiability in Re $z > 0$ uniformly for Im $z \geq 0$ or ≤ 0, since $(x \pm i 0 - n p^2)^{-k}$ is a distribution in $S'(R^3)$ for $x \neq 0$, which is weakly C^∞ in x.

At $z = 0$, $T(k, l, z)$ has a $\sqrt{z}$-singularity, if $(\bar{S}_2)$ holds. In a neighborhood N of $z = 0$, $T(z)$ is given by the power series

$$T(z) = \sum_{n=0}^\infty [T(0) (R_0(z) - R_0(0))]^n T(0) \tag{3.28}$$

which converges in the topology of $\mathfrak{L}(B(\alpha, 1/2))$. The series can be split as $\Sigma_0^{s-1} + \Sigma_s^\infty$, in such a way that the second term is C^r, if $s = s(r)$. The contribution of the first term, multiplied with a C^∞ partition of the unit subordinate to N, gives a finite sum of integrals of type (3.25) by Theorem 3.1.

In a similar way, by combining the results of FADDEEV and Theorem 3.4, we can prove maximal regularity for the 3-body amplitudes. Here the notation of section 1 is too cumbersome. Let $\mathfrak{a} = \{1, 2, 3\}$, $\alpha = (a_2) = (\{i, j\}, \{k\}) \equiv i\,j$ $(= 12, 13, 23)$, and

$$M_{\mathfrak{a}}^{\alpha\beta}(z) = V_\alpha\, \delta_{\alpha\beta} + V_\alpha\, R_{\mathfrak{a}}(z)\, V_\beta = T_\alpha(z)\, \delta_{\alpha\beta} + T_\alpha(z)\, R_0(z) \sum_{\gamma \neq \alpha} M_{\mathfrak{a}}^{\gamma\beta}(z)\,. \tag{3.29}$$

Here the $T_\alpha(z)$ are the exact 2-body amplitudes transformed from their CM system into the CM system of $\mathfrak{a}$ by (1.26). Let ${}^kM_{\mathfrak{a}}^{\alpha\beta}(z)$ be (3.29) minus its first k iterations.

Theorem 3.5: Assume $v_{ij} \in B(\theta, \infty)$, $\theta > 3/2$, and $(\bar{S}_2)$. Then $A^5(z)$ can be extended to a bounded operator $B(\alpha, \beta) \to B(\gamma, \delta)$ in a scale of component spaces with $3/2 < \alpha < \gamma$, $0 < \beta < \delta$. The inhomogeneity in

$${}^4M_{\mathfrak{a}}^{\alpha\beta}(z) = {}^4\tilde{M}_{\mathfrak{a}}^{\alpha\beta}(z) + \sum_{\gamma \neq \alpha} T_\alpha(z)\, R_0(z)\, {}^4M_{\mathfrak{a}}^{\gamma\beta}(z) \tag{3.30}$$

has components in $B(\gamma, \delta)$. Assume $(\bar{S}_3)$ for the F-Y equations (3.30) in $B(\alpha, \beta)$ and let $E_{\mathfrak{a}}^n$ be the finitely many singular values with $E_{\mathfrak{a}}^n < E_{\mathfrak{a}}^c$. Then there exists a solution to (3.30):

$${}^4M_{\mathfrak{a}}^{\alpha\beta}(k, l, z) = \sum_n \frac{\varphi_{\mathfrak{a}}^{n\,\alpha}(k)\, \varphi_{\mathfrak{a}}^{n\,\beta}(l)^*}{z - E_{\mathfrak{a}}^n} + {}^4\hat{M}_{\mathfrak{a}}^{\alpha\beta}(k, l, z) \tag{3.31}$$

where

(a) $\varphi_{\mathfrak{a}}^{n\,\alpha}(k) = (V_\alpha\, \psi_{\mathfrak{a}}^n)\,(k)$ is C^∞ in k and decreases with all derivatives as $N(k, \alpha)$.

(b) ${}^4\hat{M}_{\mathfrak{a}}^{\alpha\beta}(k, l, z)$ has HC components decreasing as $N(k, \alpha)\; N(l, \alpha)$, when multiplied with $(1 + E(k))^{-1}$ or $(1 + E(l))^{-1}$, uniformly for $\mathrm{Re}\, z \leq E_0$, $\mathrm{Im}\, z \geq 0$ or ≤ 0.

(c) For sufficiently large k, the Hölder norms of ${}^kM_{\mathfrak{a}}^{\alpha\beta}(k, l, z)$ are bounded by $c\,(1 + |\,z\,|)^{-\delta}$ $(\delta > 0$, uniform convergence of an iterative solution for large $|\,z\,|)$. ${}^kM_{\mathfrak{a}}^{\alpha\beta}(k, l, z)$ is C^∞ in k, l and $\mathrm{Re}\, z$ for all $\mathrm{Im}\, z \leq 0$ or ≥ 0, except for the thresholds $(\alpha = i\,j$, $\beta = f\,g$, $\{e, f, g\} = \{h, i, j\} = \{1, 2, 3\})$

$$\begin{aligned} &z = 0, \qquad z = E_{\mathfrak{a}}^n,\; z = E_\gamma^m \quad (\gamma = 12, 13, 23) \\ &z = n_h\, k_h^2 + n_i\, n_j/(n_i + n_j)\; (k_i + k_j)^2\; (+\, E_{ij}^m) \\ &z = n_e\, l_e^2 + n_f\, n_g/(n_f + n_g)\; (l_f + l_g)^2\; (+\, E_{fg}^m)\,. \end{aligned} \tag{3.32}$$

Proof: The boundedness of $A^5(z) : B(\alpha, \beta) \to B(\gamma, \delta)$ follows easily from Theorem 3.1. A typical graph is represented in Figure 1, the vertical connections being exact 2-body amplitudes. For the pure T_2-terms, r sufficiently large, our perturbation theoretic argument needs no modification. By direct estimates, however, one sees that one acquires uniform γ-decrease already for A^5 (instead of $M = 4\,(N - 1) = 8$). The T_1-terms can be incorporated, since the singular z-dependence is through the same exponential representation as used for the propagators. Also the bound state denominators are harmless, since they never vanish together with one of the propagators on either side.

Maximal regularity will be systematically studied in section 4. The algorithm of Theorem 4.1 can be applied using a standard technique, which will be illustrated here for the typical example of Figure 1 in the equal mass case ($n_i = 1$). The amplitudes T_{12} and T_{23} at both ends of the graph lead to multiplicative singularities for

$$z = E_{12}^n + (k_1 + k_2)^2/2 + k_3^2 = E_{12}^n + 3/2\, k_3^2 \qquad z = E_{23}^m + 3/2\, l_1^2 . \tag{3.33}$$

We claim that away from (3.33) one can absorb all other singularities by partial integration, except for

$$z = 3/2\, k_3^2 , \qquad z = 3/2\, l_1^2 \qquad z = 0 , \qquad z = E_{ij}^n \qquad (\text{for any } ij, n) . \tag{3.34}$$

Assume typically $z = E + i\,\varepsilon$, $\varepsilon \geq 0$, $E < 3/2\, k_3^2$, $3/2\, l_1^2$, $E \neq 0$, E_{ij}^n. Then we choose loop momenta p_1, p_2, p_3 as in Figure 1. We have to show that the amplitude is C^∞ in k, l, E uniformly for $\varepsilon \geq 0$. Let first $\varepsilon > 0$. Then differentiation can be interchanged with the p-integration. The numerator functions are C^∞ for $\varepsilon \geq 0$ by Theorem 3.4, while differentiation increases the power of the denominators. With $p_0 = k_3$, $p_4 = l_1$ and bound state energies E_i we have to estimate

$$\int d^9 p\, \psi(p, z) \prod_{i=1}^{4} (z - E(p_{i-1}, p_i))^{-r_i} \prod_{i=1}^{3} (z - E_i - 3/2\, p_i^2)^{-s_i} . \tag{3.35}$$

All denominators are dangerous only in a bounded region K, to which we restrict (3.35) by a C^∞ partition of the unit. If we replace p_i by $p_i\,(1 - i\,\varphi)$, $1 \leq i \leq 3$, we obtain an integral $I(\varphi)$. By Lemma 4.2 the denominators never vanish for $\varepsilon > 0$ and $0 \leq \varphi \leq \varphi_0$ (φ_0 sufficiently small), or $\varepsilon \geq 0$ and $0 < \varphi \leq \varphi_0$. There exist some $c > 0$ such that for all p_i, $1 \leq i \leq 3$, and for $\varepsilon \geq 0$ the integrand is bounded by φ^{-t}, $t = \Sigma\, r_i + \Sigma\, s_i$. (3.35) is the limit $\varphi \downarrow 0$ of $I(\varphi)$ which has the integral representation

$$I(\varphi) = \sum_{m=0}^{n-1} \frac{(\varphi - \varphi_0)^m}{m!} \left.\frac{\partial^m I}{\partial \varphi^m}\right|_{\varphi_0} + \int_{\varphi_0}^{\varphi} d\varphi_1 \ldots \int_{\varphi_0}^{\varphi_{n-1}} d\varphi_n \frac{\partial^n I(\varphi_n)}{\partial \varphi_n^n} . \tag{3.36}$$

By partial integration one can absorb the φ-derivatives into ψ and obtain with $n > t$ uniform boundedness for $\varphi \downarrow 0$ [19].

Before proceeding to $N = 4$ we have to characterize the kernel of the resolvent $R_{\mathfrak{a}\mathfrak{a}'}(z)$ for the disjoint subsystems $\mathfrak{a} = \{1, 2\}$, $\mathfrak{a}' = \{3, 4\}$ (see (1.27)). For $\alpha, \beta \in \{12, 34\}$

$$M_{\mathfrak{a}\mathfrak{a}'}^{\alpha\beta}(z) = V_\alpha\, \delta_{\alpha\beta} + V_\alpha\, R_{\mathfrak{a}\mathfrak{a}'}(z)\, V_\beta = T_\alpha(z)\, \delta_{\alpha\beta} + T_\alpha(z)\, R_0(z) \sum_{\gamma \neq \alpha} M_{\mathfrak{a}\mathfrak{a}'}^{\gamma\beta}(z) . \tag{3.37}$$

Let ${}^k M_{\mathfrak{a}\mathfrak{a}'}^{\alpha\beta}(z)$ be the remainder of (3.37) after subtraction of the first k iterations.

Theorem 3.6: Assume $v_{ij} \in B(\theta, \infty)$ and $(\overline{S}_2)$. Then the kernels of ${}^r M_{\mathfrak{a}\mathfrak{a}'}^{\alpha\beta}(z)$, $r \geq 4$, in the relative coordinates of $\mathfrak{a}$, $\mathfrak{a}'$ ($k_1 + k_2 = \ldots = l_3 + l_4 = 0$) satisfy similar estimates as in Theorem 3.5: HC in k, l, z of index $\beta > 0$, and uniform decrease in k, l with exponent $\alpha > 3/2$ after multiplication with $(1 + E(k))^{-1}$ or $(1 + E(l))^{-1}$ for $\mathrm{Re}\, z$ bounded from above and $\mathrm{Im}\, z \leq 0$ or ≥ 0; for $r \geq r_0$, the Hölder norms are bounded by $c\,(1 + |\,z\,|)^{-\delta}$, $\delta > 0$, and ${}^r M$ is C^∞ in k, l, z except on the thresholds

$$\begin{aligned} &z = 0 , \qquad z = E_{12}^m , \qquad z = E_{34}^n , \qquad z = E_{12}^m + E_{34}^n \\ &z = n_1\, k_1^2 + n_2\, k_2^2\, (+\, E_{34}^n), \ \ldots, \\ &z = n_3\, l_3^2 + n_4\, l_4^2\, (+\, E_{12}^m) . \end{aligned} \tag{3.38}$$

Proof: The regularity properties follow most easily from representing ${}^4M$ as ${}^4M + \ldots {}^9M$ plus a remainder, which is a 2-body T-amplitude or the convolution X of two 2-body T-amplitudes sandwiched between two ${}^4M$-terms, ${}^4M\, R_0\, X\, R_0\, {}^4M$. The regularity of ${}^rM$, $r \geq 4$, follows from Theorems 3.1 and 3.4 by a trivial improvement of Lemma 3.3. The singular integrals for the remainder are harmless, as there are four singular denominators and four 3-dimensional integrations.

Under the assumption $v_{ij} \in B(\theta, \infty)$, $(\bar{S}_2)$, $(\bar{S}_3)$, we shall now "solve" the 4-body problem. All information about the subsystems is contained in Theorems 3.4, 3.5, 3.6. There are two types of sequences α (coupling schemes) which characterize the connectivity of the amplitudes T^α:

$$\begin{aligned} \alpha &= (\{\{i\,j\,k\}, \{l\}\}, \{\{i\,j\}, \{k\}, \{l\}\}) \equiv i\,j,\, i\,j\,k \\ \alpha &= (\{\{i\,j\}, \{k\,l\}\}, \{\{i\,j\}, \{k\}, \{l\}\}) \equiv i\,j,\, k\,l\,. \end{aligned} \tag{3.39}$$

In this notation the F-Y equations become

$$\begin{aligned} T^{ij,ijk} &= \tilde{T}^{ij,ijk} + A^{ij,ij}_{ijk}(T^{ij,ijl} + T^{ij,kl}) \\ &\quad + A^{ij,ik}_{ijk}(T^{ik,ikl} + T^{ik,jl}) + A^{ij,jk}_{ijk}(T^{jk,jkl} + T^{jk,il}) \\ T^{ij,kl} &= \tilde{T}^{ij,kl} + A^{ij,ij}_{ij,kl}(T^{ij,ijk} + T^{ij,ijl}) \\ &\quad + A^{ij,kl}_{ij,kl}(T^{kl,ikl} + T^{kl,jkl})\,, \end{aligned} \tag{3.40}$$

where (see (1.22)

$$A^{\cdot,ij}_{ijk} = (M^{\cdot,ik}_{\{ijk\}} + M^{\cdot,jk}_{\{ijk\}})\, R_0 \qquad A^{\cdot,ij}_{ij,kl} = M^{\cdot,kl}_{\{ij\}\{kl\}}\, R_0\,. \tag{3.41}$$

For the component space (2.11) the multiplicative singularities of the kernels are obtained from Theorems 3.5, 3.6. The singular factors on the left of T^α are typically ($n_i \equiv 1$, $\Sigma\, k_i = 0$):

$$\begin{aligned} T^{12,123}_4(k, l, z) &+ \sum_n \frac{\varphi^n_{12}\,(k_1 - k_2)\; T^{12,123}_{3,n}\,(k_1 + k_2 - 2\,k_3,\, k_4,\, l,\, z)}{z - E^n_{12} - (k_1 + k_2 - 2\,k_3)^2/6 - 4/3\,k_4^2} \\ &+ \sum_m \frac{\varphi^m_{12,123}\,(k_1 - k_2,\, k_1 + k_2 - 2\,k_3)\; T^{12,123}_{2,m}(k_4,\, l,\, z)}{z - E^m_{123} - 4/3\,k_4^2} \end{aligned} \tag{3.42}$$

where $\varphi^n_{ij} = V_{ij}\,\psi^n_{ij}$, $\varphi^m_{ij,ijk} = V_{ij}\,\psi^m_{ijk}$.

We claim that $A^{12}(z)$ has components which are HC and of uniform decrease, as required for the compactness of the F-Y kernel for $\operatorname{Im} z \geq 0$ or ≤ 0. For the decrease at infinity in the external momenta one has to study the pure ϱ_2-terms. Instead of potentials there are now exact 2-, 3- and 2-2-body amplitudes to be introduced into Lemma 3.3. This does not present any difficulty, since the highly connected remainders have much better asymptotic properties. In the study of the local singularities we shall again use the method of Theorem 3.1. A^{12} is of the form $\Pi^{12}_{i=1}\,(M_{a(i)}\,R_0)$ with partitions $a(i)$ into $\{i\,j\,k\}, \{l\}$ or $\{i\,j\}, \{k\,l\}$. We split every M_a as

$$M_a = \sum_{r=0}^{3} A^r_a\, \tilde{M}_a + A^4_a\, M_a = \sum_{r=0}^{3} \tilde{M}_a (A^T_a)^r + M_a (A^T_a)^4\,. \tag{3.43}$$

Here A_a is the F-Y kernel for the subsystem a, therefore again a product of 2-body amplitudes. Hence A^{12} is a finite sum of terms of the following type:

(a) Products of 2-body amplitudes. Then Theorem 3.1 can be directly applied.

(b) One highly connected part of some M_a as factor. In such a term there must occur the typical factor

$$\prod_{i=1}^{6} (A^{r(i)}_{a(i)} \tilde{M}_{a(i)} R_0) A^4_{a(7)} M_{a(7)} \qquad (0 \leq r(i) \leq 3) . \tag{3.44}$$

Theorem 3.1 shows that (3.44) has already the desired regularity properties. With only an infinitesimal loss in the Hölder index the remaining factors to the right and left can be multiplied, using Privalov's lemma repeatedly.

(c) At least two factors of the type M_a occur. Then one has in this product a factor

$$M_{a(1)} A^{T4}_{a(1)} R_0 \prod_{j=2}^{k-1} (A^{r(j)}_{a(j)} \tilde{M}_{a(j)} R_0) A^4_{a(k)} M_{a(k)} \tag{3.45}$$

with possibly $k = 2$. Again the regularity properties follow already from Lemma 3.2 and 3.3.

Thus the 4-body F-Y equations make sense in a scale $B(\gamma, \delta) \supset B(\alpha, \beta)$, and the Fredholm alternative applies. By repeating the discussion in [3] one can show that the set of singular values, S, is countable and of the form $S = \{z = E \pm i\,0\}$ with possible accumulation points at most at the thresholds E_{A_2} for two fragments. For $E \pm i\,0 \notin S$, one can define $\Phi_A(f)\,(E \pm i\,0, k)$, which satisfies (R). By relating $\Phi_A(f)$ (z, k) to the spectral measure of H as in section 2, one proves the square integrability of $\Phi_A(f)\,(E_A(k) \pm i\,0, k)$ and the identity

$$(f, g) = \sum_{i>1} (f, \Omega^{ex}_{A_i} (\Omega^{ex}_{A_i})^* g) + (f, Pg) \tag{3.46}$$

on a dense set of f, g, where P is the projector on the discrete spectrum of H. Leaving a proof of maximal regularity as an exercise we have obtained the

Theorem 3.7: Assume $v_{ij} \in B(\theta, \infty)$ and $(\bar{S}_M)$ for all 2- and 3-particle subsystems. Then the 4-body system is asymptotically complete.

§ 4. Unitarity and Maximal Analyticity

In this section we shall study not too singular 2-body potentials $V_{ij}(x)$, which decrease exponentially in x-space. More precisely, we require that $v_{ij}(p)$ belong to the class $H(\theta, \varrho)$

$$H(\theta, \varrho) = \{v(p) = v(-p^*)^* \quad \text{holomorphic for} \quad |\operatorname{Im} p| \leq \varrho ,$$
$$\sup\,(1 + |p|)^\theta\, |D\, u(p)| < \infty\} \tag{4.1}$$

where $\theta > 3/2$, $\varrho > 0$, D any differential monomial and sup extended over all $|\operatorname{Im} p| \leq \varrho$.

In generalized perturbation theory with holomorphic numerator functions, the Feynman integrals are holomorphic for real momenta and $\operatorname{Im} z \to 0$, except on certain Landau varieties L_i. Let us assume that there are only finitely many channels by imposing (S). Then we expect that the exact N-body amplitudes satisfy maximal analyticity (MA) in the form:

(MA): There are only finitely many Landau varieties L_i in the physical region. The N-body resolvents are holomorphic in the physical region except on $\bigcup L_i$.

We shall prove MA for purely repulsive potentials (see section 5) in $H(\theta, \varrho)$, that is only for one-channel systems. Using recent results by FEDERBUSH [20], one should be able to extend our results at least to multichannel systems satisfying $(\overline{S}_M)$ for all $M \leq N$. However, a complete argument will go far beyond the scope of our present exposition.

MA will be used recursively to prove unitarity (asymptotic completeness) according to the following scheme: Assume that one can "solve" the N-body problem in some $B(\alpha, \beta)$ in the spirit of FADDEEV (see section 2). Then in a neighborhood of any point z_0, where $(1 - A^k(z_0))^{-1}$ exists and where $A^k(z)$ is bounded in $B(\alpha, \beta)$ and in z HC in the strong topology of $\mathfrak{L}(B(\alpha, \beta))$, $(1 - A^k(z))^{-1}$ exists as a convergent series in $\mathfrak{L}(B(\alpha, \beta))$:

$$(1 - A^k(z))^{-1} = (1 - A^k(z_0))^{-1} \sum_{n=0}^{\infty} [(A^k(z) - A^k(z_0))\,(1 - A^k(z_0))^{-1}]^n . \tag{4.2}$$

Assume that all subamplitudes satisfy MA. Then we shall show that around every point z_0, where the homogeneous F-Y equation $f = A(z_0) f$ has no non-trivial solution, (4.2) converges for some k and allows to prove MA for the N-body resolvent. These analyticity properties can then be used (if $(\overline{S}_N)$ holds) to prove compactness of some power of the $(N+1)$-body kernel, by a deformation of the contour in the multiple singular integrals. By this method one only retains threshold singularities, which can be easily estimated [27].

As in section 3 we try to prove that qualitative properties of perturbation theory hold for the exact amplitudes. Therefore we shall first investigate MA in perturbation theory for potentials in $H(\theta, \varrho)$.

A graph G in the Born series $V_a + \Sigma\, V_a\,(R_0(z)\, V_a)^n$ for $T_a(z)$ (a: partition of $\{1, \ldots n\}$) is called c-connected, if c is the largest integer such that by cutting G at $c - 1$ intermediate states one obtaines c subgraphs with all particles in each set of a connected.

Theorem 4.1: Assume $v_{ij} \in H(\theta, \varrho)$. Let G be a c-connected graph for $T(z)$ with left- and right connectivity $\alpha_1 = (a_1, \ldots a_{N-1})$, $\beta_1 = (b_1, \ldots n_{N-1})$. There exists a c_0 (depending only on $m_1, \ldots m_N$) such that, if $c \geq c_0$, the Feynman amplitude $G(k, l, z)$ of G is holomorphic for $k, l \in \mathbf{R}^{3N}$ and $\operatorname{Im} z \to 0$, except on the thresholds

$$z = E_{a_i}(k), \qquad z = E_{b_i}(l) \qquad (1 \leq i \leq N-1) . \tag{4.3}$$

Proof: Let $G = V^1\, \Pi_{\varkappa=2}^{K}\,(R_0(z)\, V^\varkappa)$, where $V^\varkappa = V_{i(\varkappa)\, j(\varkappa)}$. We denote the external momenta by $q^0 = k$, $q^k = l$ and the internal momenta between $V^\varkappa$ and $V^{\varkappa+1}$ by $q^\varkappa$, $1 \leq \varkappa \leq K-1$. One has, by momentum conservation, for $1 \leq \varkappa \leq K$

$$q_k^{\varkappa-1} = q_k^\varkappa \qquad k \neq i(\varkappa), j(\varkappa) \qquad q_{i(\varkappa)}^{\varkappa-1} + q_{j(\varkappa)}^{\varkappa-1} = q_{i(\varkappa)}^\varkappa + q_{j(\varkappa)}^\varkappa . \tag{4.4}$$

We shall construct for a highly connected G a particular solution $\overline{q}$ of (4.4) which satisfies $\overline{q}^\lambda = 0$ for some $1 \leq \lambda \leq K-1$. Then $\overline{q}_i^\varkappa$, $0 \leq \varkappa \leq \lambda$, will be linear combinations of k, and the $\overline{q}_i^\varkappa$, $\lambda < \varkappa \leq K$, of l.

The general solution $\tilde{q}^\varkappa$ of the homogeneous equation ($\tilde{q}^0 = 0 = \tilde{q}^K$) can be parametrized by L loop momenta $p_l, \ldots p_L \in \mathbf{R}^3$, where $L = K + 1 - N$, if G is connected. We take as loop momenta a consistent choice of particle momenta of the following type: from the left we cut the graph G once it is 1-, 2-, ... c-connected and choose as loop momenta each time $N - 1$ particle momenta. Within the 1-connected components we again cut highly connected subsystems. Inductively we can find loop momenta such that every particle momentum is a linear combination of at most $2N$ of the $p_1, \ldots p_L$, all with coefficients ± 1.

For every intermediate state $q^\varkappa$ there is a denominator $D^\varkappa = (z - E(q^\varkappa))$. Let $\operatorname{Re} z = E$, $\operatorname{Im} z = \varepsilon \geq 0$. Consider the "rotation" $p_i \to p_i (1 - i\varphi)$, $0 \leq \varphi \leq \varphi_0$. We shall choose the $\bar{q}^\varkappa$ in such a way that $E + i\varepsilon \neq (\bar{q}^\varkappa + \tilde{q}^\varkappa (1 - i\varphi))$ holds for all $\varepsilon \geq 0$ and $0 \leq \varphi \leq \varphi_0$, except on the thresholds (4.3). The proof of the following lemma can be found in App. B:

Lemma 4.2: If G is c-connected, $c \geq c_0$, then there exists a solution $\bar{q}^\varkappa$ of (4.4) satisfying:

(1) $\bar{q}^\varkappa$ is a linear combination of $k_1, \ldots k_N$, $\bar{q}^\varkappa = k^\varkappa$, if $0 \leq \varkappa < \lambda$; $\bar{q}^\lambda = 0$; $\bar{q}^\varkappa = l^\varkappa$ is a linear combination of $l_1, \ldots l_N$, if $\lambda < \varkappa \leq k$.

(2) For every real solution $\tilde{q}^\varkappa$ of (4.4) with $\tilde{q}^0 = \tilde{q}^k = 0$, $E + i\varepsilon \neq E(\bar{q}^\varkappa + \tilde{q}^\varkappa (1 - \varphi))$ for all $\varepsilon > 0$ and $0 \leq \varphi \leq \varphi_0$, $\varphi_0 \geq 0$ sufficiently small. For $\varepsilon \downarrow 0$ and $0 < \varphi \leq \varphi_0$, the equality sign holds only if $\tilde{q}^\varkappa = 0$ and if E, k, l satisfy (4.3).

(3) $\bar{q}^\varkappa$ can be chosen holomorphic in E, k and l, except on (4.3), and everywhere HC.

(4) After identification of two solutions $'\bar{q}^\varkappa$, $''\bar{q}^\varkappa$ for two graphs G', G'' which differ only in the repetitive occurrence of some $\bar{q}^\varkappa$, (1) ,(2) and (3) can be accomplished using only a finite number of different classes of solutions.

Lemma 4.2 allows to avoid the singularities of the propagators for $\operatorname{Im} z \to 0$ by a deformation of the contour of loop momenta $(p) \to (1 - i\varphi)(p)$. However, we must take into account the region of analyticity of the potentials and the behaviour of the integrands at infinity.

Lemma 4.3: Let G be a graph with loop momenta $(p) = (p_1, \ldots p_L)$ as before. For every $\lambda > 0$ there is a $c < \infty$ and a 1-parameter family $\Gamma(\varphi)$ of contours in the $\mathbf{C}^{3L}$ of complex loop momenta with

(1) $\Gamma(\varphi)$ is C^∞ and semiflat, i.e. in the natural direct product $C^{3L} = (\operatorname{Re} C^{3L}) \times (\operatorname{Im} C^{3L})$ $\Gamma(\varphi)$ is a C^∞ cross section over $\operatorname{Re} C^{3L}$.

(2) Let $p^\varkappa$ be the combination of loop momenta $(p) \in \Gamma(\varphi)$ in the intermediate state $\varkappa$. If $E(\operatorname{Re} p^\varkappa) < \lambda$, then $\operatorname{Im} p^\varkappa = -\varphi \operatorname{Re} p^\varkappa$.

(3) On $\Gamma(\varphi)$, $|\operatorname{Im} p_t| \leq c\,|\varphi|$ for $1 \leq t \leq L$.

Proof: Let $(d) = (d_1, \ldots d_{K-1})$, $d_\varkappa = 0{,}1$ $(1 \leq \varkappa \leq K - 1)$. A covering $\{D(d)\}$ of R^{3L} is defined by

$$D(d) = \{(p) \in R^{3L}: E(p^\varkappa) < \lambda + 1, \quad \text{if} \quad d_\varkappa = 0; \qquad E(p) > \lambda, \quad \text{if} \quad d_\varkappa = 1\}. \qquad (4.5)$$

Let $\{\Phi(d)\}$ be a partition of the unit in R^{3L} subordinate to $\{D(d)\}$. For every (d), one can complete the $\{p_i^\varkappa : d_\varkappa = 0, 1 \leq i \leq N\}$ by a minimal set of loop momenta $\{p_1', \ldots p_r'\} \subset \{p_1, \ldots p_L\}$, such that every particle momentum p_j^λ depends on the $\{p_i^\varkappa\} \cup \{p_\varrho'\}$

via (4.4) and $p^0 = p^K = 0$. The component $\overline{p}_t$ of every loop momentum p_t along the $\{p'_\varrho\}$ is unique and C^∞ in (p).

We define $\psi(d)\,(p) = (\psi(d)\,(p)_1, \ldots \psi(d)\,(p)_L)$ by

$$\psi(d)\,(p)_t = \begin{cases} 0 \ : p_t \in \{p'_\varrho\} \\ p_t : p_t \in \{p_i^\varkappa : d_\varkappa = 0\} \\ p_t - \overline{p}_t : \text{otherwise} \end{cases} \tag{4.6}$$

$$\Gamma(\varphi) = \{(p) \in C^{3L} : \operatorname{Im} p_t = -\varphi \sum_{(d)} \psi(d)\,(\operatorname{Re} p)_t\, \Phi(d)\,(\operatorname{Re} p)\}\,. \tag{4.7}$$

Clearly $\Gamma(\varphi)$ is semiflat and C^∞. If $(p) \in \Gamma(\varphi)$ satisfies $E(\operatorname{Re} p^\varkappa) < \lambda$, then for all (d) $D(d) \ni (\operatorname{Re} p)\ d_\varkappa = 0$ and $\psi(d)\,(\operatorname{Re} p)^\varkappa = \operatorname{Re} p^\varkappa$ and thus $\operatorname{Im} p^\varkappa = -\varphi \operatorname{Re} p^\varkappa$.

Given (d) and $p_t \notin \{p'_\varrho\} \cup \{p_i^\varkappa\}$, it is easy to estimate $|\,p_t - \overline{p}_t\,|$: let $\varkappa', \varkappa''$ be the closest intermediate states to the right and left of the particle line carrying p_t with $d_{\varkappa'} = 0$ or $\varkappa' = 0$ and $d_{\varkappa''} = 0$ or $\varkappa'' = K$. Then there are $\sigma'_i, \sigma''_i \in \{0, \pm 1\}$, such that

$$p_t - \overline{p}_t = \sum_{i=1}^{N-1} (\sigma'_i\, p_i^{\varkappa'} + \sigma''_i\, p_i^{\varkappa''})\,. \tag{4.8}$$

Since on the support of $\Phi(d)$ the $p_i^\varkappa$ with $d_\varkappa = 0$ are bounded by $\sqrt{2\,m\,(\lambda+1)}$, $m = \max m_i$, one obtains for $(p) \in \Gamma(\varphi)$

$$|\operatorname{Im} p_t| \le |\varphi|\, 2\,(N-1)\,\sqrt{2m\,(\lambda+1)} \tag{4.9}$$

For fixed λ, there is a $\varphi_0 > 0$ such that, for $|\,\varphi\,| \le \varphi_0$ and $(p) \in \Gamma(\varphi)$, the argument of every $v^\varkappa(q)$ stays in $\{|\operatorname{Im} q\,| \le \varrho\}$, since q is a linear combination of $\le 4N$ loop momenta p_t with coefficients ± 1. Let $\operatorname{Re} z \le E$, $E \ge 0$ arbitrary but fixed. If $(p) \in \Gamma(\varphi)$ and $E(\operatorname{Re} p^\varkappa) < \lambda$, then by Lemma 4.2 $D^\varkappa \neq 0$ except at thresholds for $0 < |\,\varphi\,| \le \varphi_0$ and $\operatorname{sgn}\varphi \cdot \operatorname{Im} z \ge 0$. On the other hand, if $E(\operatorname{Re} p^\varkappa) \ge \lambda$, then we consider

$$\operatorname{Re} D^\varkappa = \operatorname{Re} z - E\,(k^\varkappa + l^\varkappa + \operatorname{Re} p^\varkappa) + E(\operatorname{Im} p^\varkappa)\,. \tag{4.10}$$

If $\varkappa$ is not within an annihilation scheme (see proof of Lemma 4.2) and if the connectivity of the subgraph to the left is a_i, $i > 1$, then $l^\varkappa = 0$ and

$$E\,(k^\varkappa + \operatorname{Re} p^\varkappa) = \sum_{j=1}^{i} n(\mathfrak{a}_j)\, k(\mathfrak{a}_j)^2 + \sum_{t=1}^{N} n_t \left(k_t^\varkappa - \frac{m_t}{m(\mathfrak{a}_{j(t)})}\, k(\mathfrak{a}_{j(t)}) + \operatorname{Re} p_t^\varkappa\right)^2 \tag{4.11}$$

where the $k_t^\varkappa$ are linear combinations of the $k(\mathfrak{a}_j)$ with for all graphs G uniformly bounded coefficients. Hence (4.10) never vanishes for sufficiently large λ. Otherwise $E(k^\varkappa) \le E$, which leads to the same conclusion.

For $\operatorname{Re} z \le E$, $\operatorname{Im} z \operatorname{sgn}\varphi > 0$ and sufficiently small φ_0

$$G(k, l, z) = \int_{\Gamma(0)} dp\; I(k, l, p, z) = \int_{\Gamma(\varphi)} dp\; I(k, l, p, z)\,, \tag{4.12}$$

since the integrand I vanishes at infinity typically as $\Pi\,(1 + p_i^2)^{-1} \times (1 + |\,p_i - \Sigma\, c_{ij}\, p_j\,|)^{-3/2}$, $c_{ij} = 0$ for $j < i$, if k, l, z are fixed, and since the deformation $\Gamma(\varphi')$, $0 \le |\,\varphi'\,| \le |\,\varphi\,|$, proceeds over semiflat contours within the region of analyticity. (4.12) defines an analytic continuation of $G(k, l, z)$ for $\operatorname{Im} z \to 0$ outside of (4.3) by a suitable choice of the η's in Lemma 4.2.

Theorem 4.4: Under the assumptions of Theorem 4.1 $G(k, l, z)$ is HC in k, l, z and uniformly decreasing in k, l for $\operatorname{Re} z \leq E$, $\operatorname{Im} z \geq 0$ or ≤ 0.

Proof: We shall give here a new proof of Theorem 3.1 which, however, can easily be generalized to the exact amplitudes. Consider (4.12) for $\varphi \neq 0$. The dependence of I on k, l, z is explicit in the $\bar{q}^\varkappa(k, l, \operatorname{Re} z)$ of Lemma 4.2. There is a natural decomposition of the region of p-integration by cutting at $E(\operatorname{Re} p^\varkappa) = 1$. The dominant asymptotic behaviour for large k, l comes from $\{E(\operatorname{Re} p^\varkappa) \geq 1, 1 \leq \varkappa \leq K - 1\}$, and is covered by Lemma 3.3. The strongest local singularities arise, when all $E(\operatorname{Re} p^\varkappa) \leq 1$, and then on $\Gamma(\varphi)$ $\operatorname{Im} p^\varkappa = -\varphi \operatorname{Re} p^\varkappa$. Using Lemma 3.2 one obtains

$$|D^\varkappa| \geq c\, E(\operatorname{Re} p^\varkappa) \tag{4.13}$$

where $c > 0$ is independent of (p). For fixed $\operatorname{Re} z, k, l$, the HC in $\operatorname{Im} z \geq 0$ or ≤ 0 follows very simple. If $\operatorname{Re} z \neq 0$, the singularities of the multiple singular integral decouple at an intermediate state, where $\bar{q}^\varkappa = 0$ and hence $D^\varkappa \neq 0$. Then Privalov's lemma can be repeatedly applied. In any case we can use (4.13) for estimating (4.12). We subdivide the region of integration into sectors W, where

$$E(\operatorname{Re} p^{\varkappa(1)}) \leq \ldots \leq E(\operatorname{Re} p^{\varkappa(K-1)}) \leq 1 . \tag{4.14}$$

In W we introduce as new coordinates a maximal set of $\nu(1)$ independent particle momenta $p'_1, \ldots p'_{\nu(1)}$ through $\varkappa(1)$, then $\nu(2) - \nu(1)$ through $\varkappa(2)$, etc. One obtains in W

$$c(p'_k)^2/2\,m \leq \min_{j \geqslant t} |D^{\varkappa(j)}| \tag{4.15}$$

for $1 \leq t \leq K - 1$ and $\nu\,(t-1) + 1 \leq k \leq \nu(t)$. Lemma 3.2 gives the algorithm for proving the HC of (4.12), since every d^3p' can absorb a factor $\Pi (D^{\varkappa(j)})^{\delta(j)}$, if $\Sigma\, \delta(j) < 3/2$ and if the $\varkappa(j)$ are compatible with (4.15). Since the Hölder index of $\bar{q}^\varkappa(k, l, \operatorname{Re} z)$ can be chosen independent of G, one obtains a uniform Hölder index $\bar{\mu} > 0$ for all graphs G as in (3.19), ... (3.22).

It is now rather easy to reduce the study of the exact amplitudes to perturbation theory, if there are no solutions of the homogeneous F-Y equations in $B(\alpha, \beta)$ for all $M \leq N$. This condition will be denoted by $(\tilde{S}_N)$.

Theorem 4.5: Assume $v_{ij} \in H(\theta, \varrho)$, $\theta > 3/2$, $\varrho > 0$, and $(\tilde{S}_N)$. Then the N-body T-operator has a representation $T(z) = T_1(z) + T_2(z)$. $T_1(z)$ is a product of M-body amplitudes, $M < N$. $T_2(z)$ is a solution of the F-Y equations in some $B(\alpha, \beta)$, $\alpha > 3/2$, $\beta > 0$, with a kernel which is MA except at the thresholds (4.3). The kernel of the connected part of $T_1(z)$ is holomorphic for real $k, l \in R_0^{3N}$ and $\operatorname{Im} z \to 0$ except on the union of finitely many Landau varieties L_i corresponding to physical rescattering or threshold configurations.

Proof: We use induction with respect to N. For $N = 2$, no assumptions on the solution of subprocesses are necessary, and all information about the F-Y kernel and inhomogeneity is contained in the definition of $H(\theta, \varrho)$. For $N > 2$, the exact amplitude of a subsystem a_r with connectivity α_r and β_r from both sides can be split as

$$N_{a_r}^{\alpha_r \beta_r} = {}^1N_{a_r}^{\alpha_r \beta_r} + {}^2N_{a_r}^{\alpha_r \beta_r} \tag{4.16}$$

where ${}^2N_a$ is highly connected within a. For estimating the F-Y kernel $A^M(z)$ we choose real momenta $q^\varkappa = \bar{q}^\varkappa + \bar{\bar{q}}^\varkappa$ as in Lemma 4.2. If some ${}^2N_a$-amplitude occurs, then we can assume that the critical intermediate state (B.17) or (B.19) lies on one side of ${}^2N_a$. If ${}^2N_a$ is within a dissipation scheme (B.21), then the dissipation within a makes that in the intermediate states on both sides of ${}^2N_a$ $\bar{q}^\varkappa$ and $\bar{q}^{\varkappa+1}$ have vanishing relative momenta within a. In an annihilation scheme a dissipation within a is compatible with Theorem 4.1 and produces the same effect. We obtain $q^\varkappa = \bar{q}^\varkappa + \bar{\bar{q}}^\varkappa$, where $\bar{q}^\varkappa(k, l, \operatorname{Re} z)$ can be chosen holomorphic except on thresholds and HC everywhere, if M is sufficiently large. $\bar{\bar{q}}^\varkappa$ is a linear combination of loop momenta, which in a rotation $(p) \to (1 - i\,\varphi)\,(p)$, $\operatorname{sgn}\varphi \operatorname{Im} z \geq 0$, $|\,\varphi\,| \leq \varphi_0$, never crosses a singularity of a propagator. In the intermediate states next to a ${}^2N_a$-amplitude the relative momenta within a in $\bar{q}^\varkappa$ are zero.

We make the induction assumption that all highly connected subamplitudes have the form (see (1.26))

$$ {}^2N^{\alpha_r\beta_r}_{a_r}(k, l, z) = \delta_{a_r}(k - l)\; {}^2\hat{N}^{\alpha_r\beta_r}_{a_r}\Big(k_1 - \frac{m_1}{m(\mathfrak{a}_{j(1)})}\,\bar{k}(\mathfrak{a}_{j(1)})\,,\ \ldots,\ z - E_{a_r}(k)\Big) \tag{4.17} $$

${}^2\hat{N}^{\alpha\beta}_a(\hat{k}, \hat{l}, z)$ is holomorphic for real $\hat{k}$, $\hat{l}$ (always with $\hat{k}(\mathfrak{a}_j) = \hat{l}(\mathfrak{a}_j) = 0$, $1 \leq j \leq r$) and $z \notin [0, \infty)$ and for $\hat{k} \in \Gamma_{\alpha_r}(\varphi)$, $\hat{l} \in \Gamma_{\beta_r}(\varphi)$ and $\operatorname{Re} z \leq E$, $\arg z \neq \arg\,(1 - i\,\varphi)^2$, where $|\,\varphi\,| \leq \varphi_0$ and $\varphi_0 > 0$ is sufficiently small. $\Gamma_{\alpha_r}(\varphi)$ is any semiflat C^∞ contour in the relative momenta with uniformly small imaginary parts and the restriction that, if $a_i \subset a_r$ and if $E_{a_i}(\operatorname{Re}\hat{k}) < \lambda$, then $\operatorname{Im}\hat{k}(a_{j+1}/a_j) = -\varphi \operatorname{Re}\hat{k}(a_{j+1}/a_j)$ for all $i - 1 \leq j \leq r$ (λ fixed, sufficiently large). Furthermore $\hat{N}$ has at $z = 0$ on these $\Gamma_{\alpha_r}(\varphi)$, $\Gamma_{\beta_r}(\varphi)$ HC boundary values. The asymptotic behaviour in $\hat{k}$, $\hat{l}$ should be of the form $N(\hat{k}, \alpha)$ $N(\hat{l}, \alpha)$ after multiplication with $(1 + E(\hat{k}))^{-1}$ or $(1 + E(\hat{l}))^{-1}$.

Let now M be sufficiently large (related to the minimal connectivity in Theorem 4.1) and $\operatorname{Re} z \leq E$. The ${}^2N_a$-insertions then stay well-behaved also on rotated contours. The external momenta of $A^M(k, l, z)$ only enter in the form ${}^2N^{\alpha\beta}_a(\hat{p}^\varkappa, \hat{p}^{\varkappa+1}, z - E_a(\bar{p}^\varkappa + \bar{q}^\varkappa))$. Let $\operatorname{Im} z \neq 0$ and $\operatorname{sgn}\varphi \operatorname{Im} z \geq 0$. We claim that by placing the loop momenta (p) on a C^∞ semiflat contour $\Gamma(\varphi)$ which takes into account the restrictions on the $\hat{p}^\varkappa$, $\hat{p}^{\varkappa+1}$, we stay for small φ in the holomorphy domains of the exact amplitudes. This follows from Lemma 4.2: for $\operatorname{Im} z \neq 0$ on such a contour $z - E_a(\bar{p}^\varkappa + \bar{q}^\varkappa)$ never lies on $\{y : \arg y = \arg\,(1 - i\,\varphi)^2\}$. For $\operatorname{Im} z \to 0$, the cut is reached at most at the origin and only if (4.3) holds. The assumed asymptotic behaviour guarantees that there are no contributions from infinity in the deformation $\Gamma(\varphi')$, $0 \leq \varphi' \leq \varphi$.

By Theorem 4.4, the kernel of $A^M(z)\;\mathrm{R}_0(z)^{-1}$ has uniform HC and growth properties for $\operatorname{Im} z \to 0$ and allows a solution of the N-body problem in the spirit of FADDEEV. Let us reproduce the analyticity properties of the induction! For $z_0 = E_0 + i\,0$, $E_0 \leq E$, the homogeneous equation $f = A(z_0)\,f$ has no non-trivial solution in $B(\gamma, \delta)$ by $(\tilde{S}_N)$. Assume that for all $K \geq M$, $f_K = A^K(z_0)\,f_K$ has a non-trivial solution in $B(\gamma, \delta)$. Then for some $1 \leq k \leq K - 1$

$$ A(z_0)\,g_K = \exp(2\,\pi\,i\,k/K)\,g_K \tag{4.18} $$

has a non-trivial solution in $B(\gamma, \delta)$. By varying K, one can obtain for the compact operator $A^M(z_0)$ infinitely many different eigenvalues on the unit circle, which is impossible. Hence $(1 - A^K(z_0))^{-1}$ exists on $B(\alpha, \beta)$ for some $K \geq M$ and also $(1 - A^K(z))^{-1}$ for all $| z - z_0 | \leq \delta > 0$, $\operatorname{Im} z \cdot \operatorname{Im} z_0 \geq 0$, and is given by the series (4.2).

There exists a splitting $T(z) = T_1(z) + T_2(z)$, where $T_2(z)$ is a sum of terms of the form

$$A^K(z) \sum_{n=0}^{\infty} [(1 - A^K(z_0))^{-1} (A^K(z) - A^K(z_0))]^n (1 - A^K(z_0))^{-1} A^K(z) \tilde{T}(z) \qquad (4.19)$$

We study the kernel of (4.19) and in particular a term of n^{th} order with right- and left-connectivity α, β. We try to continue analytically in k, l, z on contours $\Gamma_\alpha(\varphi)$, $\Gamma_\beta(\varphi)$ with $\operatorname{Re} z \leq E$, $\arg z \neq \arg (1 - i\varphi)^2$ and $| z - z_0 | < \delta' > 0$. K is assumed to be large. We use

$$(1 - A^K(z_0))^{-1} = A^K(z_0) + 1 + A^K(z_0) (1 - A^K(z_0))^{-1} A^K(z_0) \qquad (4.20)$$

In the first two terms one can reach contours $\Gamma_0(\varphi)$ in the external momenta, similarly in the third term, where we keep the intermediate momenta on both sides of $(1 - A^K(z_0))^{-1}$ real. The series (4.19) converges uniformly on these deformed contours for $| z - z_0 | < \delta'$, $\delta' > 0$ sufficiently small. Using the Heine-Borel theorem one obtains finitely many $z_{01} = 0 < z_{02} < \ldots < z_{0s} = E$, such that the continuations (4.19) provide the analyticity in the "second sheet" of the induction, if φ_0 is small enough. The HC of the solution $T_2(z)$ in $B(\alpha, \beta)$ leads to the HC of the boundary values of (4.17).

$T_1(z)$ can be decomposed further by splitting all subamplitudes into highly connected remainders with only threshold singularities and potentials and free propagators. This decomposition is finite by Theorem 4.1. By applying the Landau argument [13] on necessary conditions for physical region singularities to the case, where the numerator functions have threshold singularities, one proves that except on the union of finitely many real algebraic varieties of the COLEMAN-NORTON type [9], each connected with a perturbation-theoretic diagram, the connected part of $T_1(k, l, z)$ is holomorphic for $k, l \in R_0^{3N}$ and $\operatorname{Im} z \to 0$. Disconnected components have a similar behaviour on linear subspaces (see (1.25)).

The N-body scattering amplitude for a 1-channel system is given by Theorem 1.3 as

$$(k \mid S \mid l) = \delta_{a_N} (k - l) - 2\pi i \, \delta(E(k) - E(l)) \, T(k, l, E(k) + i\,0) . \qquad (4.21)$$

Let us introduce angular variables on the energy shell by

$$k_i = k\, x_i,\ l_i = k\, y_i\ (1 \leq i \leq N),\ E(k) = k^2$$
$$\Sigma x_i = \Sigma y_i = 0,\ \Sigma n_i x_i^2 = \Sigma n_i y_i^2 = 1 . \qquad (4.22)$$

If the v_{ij} are superpositions of Yukawa potentials

$$v_{ij}(p) = \int \frac{d\sigma_{ij}(x)}{x + p^2}, \qquad \int d \mid \sigma_{ij}(x) \mid < \infty \qquad \operatorname{supp} d\sigma_{ij} \subset [y, \infty) , \quad y > 0 , \qquad (4.23)$$

then the physical region regularity properties of $T(k, l, z)$ can be extended to

Theorem 4.6: Assume that v_{ij} satisfies $(\tilde{S}_N)$ and (4.23). Then the connected part of $T(kx_1, \ldots ky_N, k^2 + i0)$ is for real $x_1, \ldots y_N$ on (4.22) outside of the union of finitely many Landau varieties L_i the boundary value of a function, which is holomorphic for real $x_1, \ldots y_N$ on $\{(4.22)\} - \bigcup_i L_i$ and for k^2 in a cut place $C^1 - (-\infty, -r] - [0, \infty)$ with $r > 0$ and has holomorphic boundary values for $k^2 \to (0, \infty)$.

The proof of this dispersion relation domain in the energy k^2 has been obtained for $N = 2, 3$ by Rubin, Sugar and Tiktopoulos [21] and for general N by Riahi [5], using the Fredholm series for $T(k, l, z)$. Dedicated amateurs in analyticity can find here a rich field of non-trivial exercises.

§ 5. Conclusion

The results of sections 3 and 4 do not yet provide a proof of asymptotic completeness from first principles. Essential for the construction of section 3 was the spectrum condition $(\bar{S}_M)$ for $M = 2, 3$, while section 4 was based on the stronger requirement $(\tilde{S}_N)$. It remains to show that a non-trivial class of 2-body potentials satisfies these restrictions.

We conjecture that $(\bar{S}_N)$ holds for all distinguishible $v_{ij} \in B(\theta, r)$ with sufficiently large θ and r, where possibly some small $\tilde{v}_{ij}$ have to be added to remove bound states at thresholds. For proving $(\bar{S}_2)$, we remark that, if $v \in B(\theta, 2)$, $\theta > 3/2$, then every solution of the homogeneous equation $\varphi_m(p) = (A\ (E_m + i\ 0)\ \varphi_m)\ (p)$ is C^1 in p and vanishes for $n p^2 = E_m$ [3]. Therefore even at threshold $E_m = 0$, $\varphi_m(p)/p^2$ is L^2 and therefore a bound state. Kato [23] and Birman [24] have given very general conditions on $V(x)$ which permit only finitely many eigenvalues E_m, all with finite multiplicity and $E_m \leq 0$. Faddeev [3] has remarked that the replacement $V \to (1 + \varepsilon)\ V$, ε real, $0 < |\varepsilon| \leq \varepsilon_0$, removes any possible zero energy bound state.

For purely repulsive potentials, H has no discrete spectrum [25]: Consider a Hamiltonian in $L^2(R^m)$, $H = -\Delta + q(x)$, where the real-valued q satisfies

(A) $q \in Q_\alpha(R^m)$ for some $\alpha > 0$, i.e.

$$M_q(x) = \int_{|x-y| \leqslant 1} |q(y)|\ |x - y|^{4-m-\alpha}\ dy \tag{5.1}$$

is uniformly bounded for $x \in R^m$.

(B) for every $x \in R^m$, $x \neq 0$, there exists a radial derivative $q_r(x)$ of $q(x)$ and

$$\varepsilon^{-1}\ |q\ ((1 + \varepsilon)x) - q(x)\ | \leq q_0(x) \in Q_\beta(x) \tag{5.2}$$

holds for $0 < \varepsilon < \varepsilon_0$ and some $\beta > 0$.

We call a potential q purely repulsive, if (A) and (B) hold and $q_r(x) \leq 0$ for all $x \in R^m$, $x \neq 0$. Weidmann [25] has proved the

Theorem 5.1: If q is purely repulsive, then H does not have any eigenvalue.

There are interesting classes of 2-body potentials satisfying (A) and (B), for instance superpositions of Yukawa potentials.

Theorem 5.2: Let $v_{ij} \in H(\theta, \varrho)$, $\theta > 3/2$, $\varrho > 0$, be purely repulsive. Then there are no non-trivial solutions of the homogeneous F-Y equations (1.24) in $B(\alpha, \beta)$, $\theta > \alpha > 3/2$, $\beta > 0$.

Proof: For $N = 2$, this follows from our preceeding remark concerning $(\bar{S}_2)$. Assume that there are no non-trivial solutions in $B(\alpha, \beta)$ for all subsystems. Then the N-body amplitudes have all only one component for every α. A solution of (1.24) for $z \in [0, \infty)$ is excluded by Theorem 5.1 and section 1.

Assume that there are $f^\alpha \in B(\alpha, \beta)$ satisfying (1.24) with $z = E + i\,0$, $E \geq 0$. Then, as for $N = 3$ [3], it is a consequence of the symmetry of V on $\mathfrak{D}(H^0)$ that $f^\alpha(k) = 0$ for $E(k) = E$. Furthermore $f^\alpha(k)$ is holomorphic for real $k(a_N/a_{N-1})$, since the dependence on this variable is entirely through the first potential. The estimates in [3] carry over and show that

$$(E + i\,0 - E(k))^{-1} \sum_\alpha f^\alpha(k) \tag{5.3}$$

is a L^2 function and therefore an eigenstate of H, unless $\Sigma f^\alpha = 0$. By the F-Y equation the latter condition implies $f^\alpha = 0$ for all α.

Another class of N-body forces, for which there are no solutions of the homogeneous F-Y equations in $B(\alpha, \beta)$, has weak potentials.

Theorem 5.3: Let $v_{ij} \in H(\theta, \varrho)$, $\theta > 3/2$, $\varrho > 0$. There exists a $\lambda_0 > 0$, such that $(\bar{S}_N)$ holds for all N-body systems with potentials $\lambda_{ij}\, v_{ij}$, $-\lambda_0 \leq \lambda_{ij} \leq \lambda_0$.

Proof: From Theorem 4.4 it follows, assuming $(\tilde{S}_M)$ for $M < N$, that there are no non-trivial solutions of (1.24) for $\operatorname{Re} z \leq E$, if $-\lambda_0 \leq \lambda_{ij} \leq \lambda_i$, where $\lambda_0 = \lambda_0(E)$. More careful estimates show that the norms of $A^M(z)$ as mappings from $B(\alpha, \beta)$ to $B(\gamma, \delta)$ increase only polynomially with $\operatorname{Re} z$ and are proportional to λ_0^M. Furthermore, by increasing M and decreasing $\gamma > \alpha > 3/2$, $\delta > \beta > 0$, one can find uniform bounds independent of $\operatorname{Re} z$, proportional to λ_0^M. For $N = 3$, this is a consequence of Lemma 7.2 in [3]. Since there exists probably a more direct proof of asymptotic completeness for weak potentials along the lines of [7] and [8], we shall not bring the tedious improvements of Theorem 4.4 and Lemma 3.3.

Corollary 5.4: Let $v_{ij} \in H(\theta, \varrho)$ satisfy $(\tilde{S}_N)$. Then unitarity and maximal analyticity are stable against small perturbations in $H(\theta, \varrho)$.

This concludes our presentation of the quantum mechanical N-body problem. It is desirable to treat by the FADDEEV approach more general multichannel systems with 2-body forces of short range. Short range many-body potentials are, of course, always less singular. The FEDERBUSH technique [26] appears to be promising, if the complicated geometry of deformed contours is efficiently treated. A direct proof of (S_N) (for undistinguishable particles in the subspace of totally symmetric or antisymmetric wave functions) using a priori estimates on $-\Delta + q(x)$ would be most welcome for our physical understanding of multichannel many-body systems.

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Appendix A : Kinematics

For every $\mathfrak{a} \subseteq \{1, \dots N\}$, the following splitting of the k_i, $i \in \mathfrak{a}$, into CM and relative momenta is useful

$$\bar{k}(\mathfrak{a}) = \sum_{i \in \mathfrak{a}} k_i, \quad m(\mathfrak{a}) = \sum_{i \in \mathfrak{a}} m_i \qquad \hat{k}_i(\mathfrak{a}) = k_i - \frac{m_i}{m(\mathfrak{a})} \bar{k}(\mathfrak{a}) . \tag{A.1}$$

One has with $n_i = (2\, m_i)^{-1}$, $n(\mathfrak{a}) = (2\, m(\mathfrak{a}))^{-1}$,

$$\sum_{i \in \mathfrak{a}} n_i\, k_i^2 = n(\mathfrak{a})\, \bar{k}(\mathfrak{a})^2 + \sum_{i \in \mathfrak{a}} n_i\, \hat{k}_i(\mathfrak{a})^2 . \tag{A.2}$$

Let A be a channel with $a(A) = a = (\mathfrak{a}_1, \dots \mathfrak{a}_i)$. Let $j(m)$ satisfy $\mathfrak{a}_{j(m)} \ni m$ and define

$$\begin{aligned} E_a(k) &= \sum_{j=1}^{i} n(\mathfrak{a}_j)\, k(\mathfrak{a}_j)^2 \\ E_A(k) &= E_a(k) + \sum_{j=1}^{i} E_{\mathfrak{a}_j}^{n_j} \\ E(k) &= \sum_{r=1}^{N} n_r\, k_r^2 = E_a(k) + \sum_{r=1}^{N} n_r\, \hat{k}_r(\mathfrak{a}_{j(r)})^2 . \end{aligned} \tag{A.3}$$

Every sequence $(a_1, \dots a_N)$ of partitions with $a_i \supset a_{i+1}$, $1 \leq i \leq N-1$, defines a coupling scheme for Jacobi coordinates $k(a_1) = \Sigma\, k_r$, $k(a_2/a_1)$, $\dots$ $k(a_N/a_{N-1})$:

$$k(a_{i+1}/a_i) = \frac{m(\mathfrak{a}_e)\, \bar{k}(\mathfrak{a}_f) - m(\mathfrak{a}_f)\, \bar{k}(\mathfrak{a}_e)}{m(\mathfrak{a}_e) + m(\mathfrak{a}_f)} \tag{A.4}$$

if a_{i+1} becomes a_i by connecting $\mathfrak{a}_e$, $\mathfrak{a}_f$ to $\mathfrak{a}_e \cup \mathfrak{a}_f$. Using $n(a_{i+1}/a_i) = n(\mathfrak{a}_e) + n(\mathfrak{a}_f)$ one obtains for $E(k)$ and any $2 \leq i \leq N-1$

$$E(k) = E_{a_i}(k) + \sum_{j=1}^{N-1} n(a_{j+1}/a_j)\, k(a_{j+1}/a_j)^2 . \tag{A.5}$$

For $k, l \in R_0^{3N}$ we define

$$\delta_{a_i}(k - l) = \prod_{j=1}^{i-1} \delta\, \big(k(a_{j+1}/a_j) - l\,(a_{j+1}/a_j)\big) = \delta_{A_i}\,(k - l)\,, \quad \text{if} \quad a(A_i) = a_i . \tag{A.6}$$

Let $B_M(\theta, \mu)$, $\theta > 0$, $\mu > 0$, be the Banach space of all functions f: $R_0^{3M} \to C$ satisfying $\| f \|_{\theta, \mu, M} < \infty$, where

$$|| f ||_{\theta, \mu, M} = \sup_{h, k \in R_0^{3M}} N_M(k, \theta)^{-1} \left[| f(k) | + \frac{| f(k+h) - f(k) |}{| h |^{\mu}} \right] \tag{A.7}$$

$$N_M(k, \theta) = \sum_i \prod_{j=1}^{M-1} (1 + | k_{ij} |)^{-\theta} . \tag{A.8}$$

The summation in (A.8) extends over all $(M-1)$-tupels (k_{ij}) of partial sums

$$k_{ij} = \sum_{r=1}^{M} \sigma_{ijr}\, k_r , \qquad \sigma_{ijr} = 0, \pm 1 \tag{A.9}$$

which span $k_1, \ldots k_M$. In general we need Banach spaces

$$B_{(M)}(\theta, \mu) = \bigtimes_{r=1}^{s} B_{M_r}(\theta, \mu). \tag{A.10}$$

Without confusion we shall omit the index (M). Using a Cantor scheme one proves that for $\gamma > \alpha > 0$, $\delta > \beta > 0$, every bounded linear mapping $B(\alpha, \beta) \to B(\gamma, \delta)$ is compact in $\mathfrak{L}(B(\alpha, \beta))$.

Appendix B: Auxiliary Theorems

Proof of Lemma 3.2: We need only to estimate A^{3N-4}, since in A^M, $M > 3N - 4$, each new potential leads to an additional 3-dimensional loop integration with a $(1 + t)^{-3/2}$ bound for the new intermediate state. We set $\delta = (6N - 10)^{-1}$. A^{3N-4} is a sum of products $Q_1 R_0 \ldots Q_{3N-4} R_0$, where Q_i has the form (1.22).

Let R_0' be a propagator in $Q_1 R_0 \ldots R_0 Q_K$ with l Q's to the left (if R_0' is a factor of some Q_i, $Q_i = Q_i' R_0' Q_i''$, then Q_i' is included) and r Q's to the right (Q_i'' possibly included). Then at least

$$\min \{l, r, N - 1, K - N + 1\} \tag{B.1}$$

independent particle momenta through R_0' are undetermined by the external momenta of $Q_1 R_0 \ldots Q_k$. (Proof: In R_0', at least $\min \{l, N - 1\}$ independent particle momenta are not fixed by the external momenta to the left, since $Q_1 R_0 \ldots Q_{l-1}$ produces at least the connectivity a_{N-l+1}, which is increased to a_{N-l} by the first potential in Q_l. Furthermore, the external momenta to the right can only determine $\max \{N - 1 - r, 0\}$ particle momenta).

For every $Q' R_0 Q_1 \ldots Q_K R_0 Q''$ ($Q' R_0$, $R_0 Q''$ either 1 or the Q' or Q'' are right or left ends of some Q) we define a "niveau" scheme: Let $g = \min \{N - 1, K - N + 1\}$. The propagators in $Q_g R_0 \ldots Q_{K-g+1}$ belong to the "plateau" with "height" g, the others to the "slope" with the following heights: 0 for $Q' R_0$, $R_0 Q''$, 1 for $Q_1 R_0$, $R_0 Q_K$, $\ldots$, $g - 1$ for $Q_{g-1} R_0$, $R_0 Q_{K-g+2}$.

Let $\varrho(1) \geq \varrho(2) \geq \ldots \varrho(f)$ be the order of the intermediate states in $Q_1 R_0 \ldots Q_{3N-4}$ induced by (3.11). Assume that $\varrho(1)$ lies on the slope of $Q_1 R_0 \ldots Q_{3N-4}$, where we always choose the left-hand side for a simpler notation. Then $\varrho(1)$ belongs to some $Q_{\sigma(1)} R_0$, where $\sigma(1) < N - 1$. The number $\nu(1)$ of undertermined particle momenta through $\varrho(1)$ satisfies $\nu(1) \geq \sigma(1)$. We cut the graph at $\varrho(1)$.

The $\nu(1)$ loop momenta through $\varrho(1)$ provide a factor

$$(1 + t_{\rho(1)})^{-3\nu(1)/2}. \tag{B.2}$$

Let $\mu(1)$ be the number of intermediate states to the left of $\varrho(1)$. In this subgraph there we can place a loop momentum on one of the particle lines after $\mu(1) - \nu(1) + 1$ potentials, starting e.g. from the left, which gives a $(1 + t)^{-3/2}$ bound for $\mu(1) - \nu(1) + 1$ propagators. After $\nu(1) - 1$ potentials and on $\varrho(1)$ all momenta are determined. Here we use a factor $(1 + t_{\varrho(1)})^{-1-\delta}$ from (B.2). There remains a power

$$\tau(1) \geq \sigma(1)\,(1 - 2\,\delta)/2 \tag{B.3}$$

of proper times which are larger than those of all remaining propagators. We turn to $Q''_{\sigma(1)} R_0 Q_{\sigma(1)+1} R_0 \ldots Q_{3N-4}$. In a somewhat simplified notation, let $\varrho(2)$ be the intermediate state with largest proper time, again on the slope. Then we define as above $\nu(2)$, $\sigma(2)$, $\tau(2)$. We continue with $\varrho(3), \ldots \varrho(g)$ under the same conditions, as long as $\sigma(g) \leq 2N - 3$, and obtain a reserve power

$$\tau(g) \geq \sigma(g)\,(1 - 2\,\delta)/2\,. \tag{B.4}$$

If $g = 0$, i.e. $\varrho(1)$ on the plateau, we replace (B.4) by 0. There are three alternatives:

(1) $\sigma(g) = 2N - 3$. Then $\tau(g) \geq N - 3/2 - (2N - 3)\,\delta$, which is enough to control the $\leq N - 2$ intermediate states in $Q''_{\sigma(g)} R_0 \ldots Q_{3N-4}$, where no loop momentum is free. There we need a power $(N - 2)\,(1 + \delta)$ of large proper times, which we have:

$$\tau(g) - (N - 2)\,(1 + \delta) \geq 1/2 - (3\,N - 5)\,\delta \geq 0\,. \tag{B.5}$$

(2) $N - 2 \leq \sigma(g) < 2N - 3$. By assumption the next largest propagator, $\varrho(g + 1)$, lies on the plateau of $Q''_{\sigma(g)} R_0 \ldots Q_{3N-4}$. There are $\nu\,(g + 1)$ new loop momenta, with

$$\nu\,(g + 1) \geq 3\,N - 4 - \sigma(g) - N + 1 \tag{B.6}$$

as a consequence of (B.1). The reserve, after having majorized the subgraph to the left and $\varrho\,(g + 1)$, is

$$\geq \sigma(g)\,(1 - 2\,\delta)/2 + (2\,N - 3 - \sigma(g))\,(1 - 2\,\delta)/2 \tag{B.7}$$

which is enough by (B.5).

(3) $\sigma(g) < \min\{2N - 3, N - 2\}$. In this case $\nu\,(g + 1) = N - 1$ and the reserve increases by $(N - 1)\,(1 - 2\,\delta)/2$. One obtains two niveau schemes, on which the previous operations must be repeated. Since the total length is $3N - 4$, case (3) can only occur once.

Proof of Lemma 3.3: Without restriction we can assume that in G the right connectivity β_1 is established by the last $N - 1$ potentials

$$\prod_{\varkappa = K - N + 2}^{K} (V^{\varkappa}\,R_0)\,. \tag{B.8}$$

We use the bound $C\,(1 + E(q))^{-1}$ for the ϱ_2-terms and can determine explicitely that the external momenta $l_1 \ldots l_N$ on the right hand-side occur in the $N - 1$ propagators as $N - 1$ independent partial sums with coefficients $0, \pm 1$. Therefore one obtains for the kernel of (B.8) the trivial bound $c\,N(l, 2)$ (c not always the same constant).

A bound in terms of the external momenta of the left-hand side of Q will be generated by the repeated loop integrations in the multiplication of (B.8) with $V^1 R_0 \ldots V^{K-N+1} R_0$.

Inductively we assume as bound in the external momenta $p \in \mathsf{R}_0^{3N}$ to the left a sum of terms of the type

$$c \prod_{\substack{i=1\\ i \neq k}}^{N} (1 + |\,p_i\,|)^{-\varrho} \quad \left(\varrho < \frac{3}{2}\right). \tag{B.9}$$

After the multiplication with $V^{\varkappa} R_0$, two cases have to be distinguished:

(a) $i(\varkappa) = k$ or $j(\varkappa) = k$. This leads to the typical majorization ($\chi = 1/2\ (N-1)$)

$$\leq \int \frac{dp_1' \mid v_{12}\,(p_1 - p_1') \mid}{\left(1 + |p_1'| + \sum_{i=3}^{N} |p_i|\right)^2 (1 + |p_1'|)^{\varrho} \prod_{i=3}^{N} (1 + |p_i|)^{\varrho}}$$

$$\leq c \prod_{i=3}^{N} (1 + \mid p_i \mid)^{-\varrho}\,(1 + \mid p_1 \mid)^{-\varrho-\chi}$$

$$\times \int_0^{\infty} dx\, x^2\,(1 + x)^{\chi - \theta} \left(1 + x + \sum_{i=3}^{N} \mid p_i \mid\right)^{-2} \tag{B.10}$$

using [3] ($\theta = \theta_1 + \theta_2$, $2 > \theta > 3/2$, $\theta_i \geq 0$):

$$\int d\Omega\,(p_1') \mid v_{12}\,(p_1 - p_1') \mid \leq C\,(1 + \mid p_1 \mid)^{-\theta_1}\,(1 + \mid p_1' \mid)^{-\theta_2}. \tag{B.11}$$

The integral in (B.10) is trivially bounded by $\leq c\,\Pi_{i=3}^{N}\,(1 + \mid p \mid)^{-\chi}$. Therefore

$$\varrho \to \varrho + 1/2\,(N-1) = \varrho + \chi \tag{B.12}$$

in the induction.

(b) Otherwise one has the typical situation

$$\leq \int \frac{dp_1' \mid v_{12}\,(p_1 - p_1') \mid (1 + |p_1'|)^{-\varrho}\,(1 + |p_1' + p_3 + \dots p_N|)^{-\varrho}}{\prod_{i=4}^{N} (1 + |p_i|)^{\varrho} \left(1 + |p_1'| + |p_1' + p_3 + \dots + p_N| + \sum_{i=3}^{N} |p_i|\right)^2}. \tag{B.13}$$

We apply the inequality [3]:

$$\begin{aligned}(1 + \mid p_1 - p_1' \mid)^{-\varrho}\,&(1 + \mid p_1' + p_3 + \dots + p_N \mid)^{-\varrho}\\ &\leq c\,(1 + \mid p_1 + p_3 + \dots + p_N \mid)^{-\varrho}\\ &\times [(1 + \mid p_1 - p_1' \mid)^{-\varrho} + (1 + \mid p_1' + p_3 + \dots + p_N \mid)^{-\varrho}]\end{aligned} \tag{B.14}$$

and treat each term separately.

In the first term we use (B.14):

$$\begin{aligned}&(1 + \mid p_1 - p_1' \mid)^{-\chi} \left(1 + \mid p_1' \mid + \mid p_1' + p_3 + \dots + p_N \mid + \sum_{i=3}^{N} \mid p_i \mid\right)^{-2}\\ &\leq c\,(1 + \mid p_1 + p_3 \dots + p_N \mid)^{-\chi} \prod_{i=4}^{N} (1 + \mid p_i \mid)^{-\chi}\,(1 + \mid p_3 \mid)^{-\varrho-\chi}\\ &\times (1 + |p_1'|)^{-2+\varrho+(N-1)\chi}\,[(1 + |p_1 - p_1'|)^{-\chi} + (1 + |p_1' + p_3 + \dots + p_N|)^{-\chi}]\end{aligned} \tag{B.15}$$

and the remaining p_1'-integral is uniformly bounded in $p_1, \dots p_N$, since $\theta - \chi + \varrho + \chi + 2 - \varrho - (N-1)\,\chi > 3$. The second term can be treated in the same way. Since $\mid p_2 \mid = \mid p_1 + p_3 + \dots p_N \mid$, we obtain again (B.12).

After $3N - 3$ steps we obtain as left bound $c\,N(k, \bar{\theta})$, where some $\bar{\theta} > 3/2$ can be found, since in all estimates we have not completely exploited $\theta > 3/2$.

Remark: We have used repeatedly that for $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma > 3$, the function

$$F(b, c) = \int d^3a\,(1 + \mid a \mid)^{-\alpha}\,(1 + \mid a - b \mid)^{-\beta}\,(1 + \mid a - c \mid)^{-\gamma} \tag{B.16}$$

is uniformly bounded for $(b, c) \in \mathbf{R}^6$. This can be proved by a (b, c)-dependent subdivision of the region of integration.

Proof of Lemma 4.2: The determination of $\bar{q}^\varkappa$, $0 \leq \varkappa < \lambda$, $\bar{q}^\lambda = 0$, satisfying (4.4) follows different schemes depending on the left-connectivity α_1 of G and on $E \in \mathbf{R}^1$, $k \in \mathbf{R}_0^{3N}$. Let $1 = \varkappa(1) < \varkappa(2) < \ldots < \varkappa\,(N-1)$ be such that $V^{\varkappa(p)}$ connects a_{N-p+1} to a_{N-p}, $1 \leq p \leq N-1$.

Let $0 < \eta_1, \ldots \eta_{N-1} \leq \eta$ be arbitrary, but fixed, with η to be restricted later.

(a) Assume

$$E - E_{a_2}(k) + \eta_2\, k(a_2/a_1)^2 \leq 0\,. \tag{B.17}$$

Then we shall define $\bar{q}^\varkappa$ by an "a_1-annihilation scheme", as the unique solution of (4.4) with $\bar{q}^0 = k$, $\bar{q}^{\varkappa(N-1)} = 0$ and satisfying for $1 \leq g \leq N-2$:

$$\bar{q}^{\varkappa(g)} = \bar{q}^{\varkappa(g)+1} = \ldots = \bar{q}^{\varkappa(g+1)-1}\,. \tag{B.18}$$

All $\bar{q}^\varkappa$, $1 \leq \varkappa < \varkappa\,(N-1)$, are linear combinations of the CM momenta $\bar{k}(\mathfrak{a}_j)$, $1 \leq j \leq r$, with coefficients ± 1 or 0, if $a_r = (\mathfrak{a}_1, \ldots \mathfrak{a}_r)$ is the connectivity of $V^1 \Pi_{\varrho=2}^{\varkappa} (R_0 V^\varrho)$.

(b) Assume for some $2 \leq p \leq N-1$

$$E - E_{a_p}(k) + \eta_p\, k(a_p/a_{p-1})^2 > 0$$
$$E - E_{a_{p+1}}(k) + \eta_{p+1}\, k(a_{p+1}/a_p)^2 \leq 0 \qquad (\text{if } p < N-2)\,. \tag{B.19}$$

Then we determine $\bar{q}^\varkappa$, $0 \leq \varkappa \leq \varkappa\,(N-p)$, by an "$a_p$-annihilation scheme", starting with $\bar{q}^0 = k$ and terminating with

$$\bar{q}_i^{\varkappa(N-p)} = \frac{m_i}{m(\mathfrak{a}_{j(i)})}\, \bar{k}(\mathfrak{a}_{j(i)}) = \ldots\, \bar{q}_i^{\varkappa(N+1-p)-1} \tag{B.20}$$

where $a_p = (\mathfrak{a}_1, \ldots \mathfrak{a}_p)$ and $\mathfrak{a}_{j(i)} \ni i$. There is again a unique solution to (4.4), if one requires (B.18) for $1 \leq g < N-p$. Let $1 \leq \lambda < \varkappa\,(N-p)$, and let $V^1\, \Pi_{\varrho=2}^{\lambda} (R_0\, V^\varrho)$ have connectivity a_r $(p+1 \leq r < N)$. Then $\bar{q}_i^\lambda$ is a linear combination of $(m_i/m(\mathfrak{a}'_{j(i)}))$ $\bar{k}(\mathfrak{a}'_{j(i)})$ $(a_r = (\mathfrak{a}'_1, \ldots \mathfrak{a}'_r),\ \mathfrak{a}'_{j(i)} \ni i)$ and $k(a_r/a_{r-1}), \ldots, k(a_{p+1}/a_p)$ with coefficients which are uniformly bounded for all graphs G. The $\bar{q}^\lambda$, $\lambda \geq \varkappa\,(N+1-p)$, are to be determined by a "dissipation scheme":

$$q_i^\lambda = \begin{cases} q_i^{\lambda-1} & i \neq i(\lambda), j(\lambda) \\ (q_{i(\lambda)}^{\lambda-1} + q_{j(\lambda)}^{\lambda-1})\, n_{j(\lambda)}/(n_{i(\lambda)} + n_{j(\lambda)}) & i = i(\lambda) \\ (q_{i(\lambda)}^{\lambda-1} + q_{j(\lambda)}^{\lambda-1})\, n_{i(\lambda)}/(n_{i(\lambda)} + n_{j(\lambda)})\,. & j = j(\lambda) \end{cases} \tag{B.21}$$

This scheme has the property [20] that, if $V^{\varrho+1} R_0 \ldots V^\sigma$ is connected, with external $\bar{q}^\varrho$, $\bar{q}^\sigma$ and internal momenta $\bar{q}^{\varrho+1}, \ldots \bar{q}^{\sigma-1}$ related by (B.21), then $E(\bar{q}^\sigma) \leq \chi\, E(\bar{q}^\sigma)$ for some $\chi < 1$ (χ universal for all N-body graphs with masses $m_1, \ldots m_N$). More generally, if $V^{\varrho+1} R_0 \ldots V^\sigma$ has the connectivity a, then $E_a(\bar{q}^\sigma) = E_a(\bar{q}^\varrho)$ and

$$E(\bar{q}^\sigma) - E_a(\bar{q}^\sigma) \leq \chi\big((E(\bar{q}^\varrho) - E_a(\bar{q}^\varrho)\big)\,. \tag{B.22)}$$

If $V^{\varrho+1} R_0 \ldots V^\sigma$ is sufficiently connected, then the relative momenta $\bar{q}^\sigma(a_N/a_{N-1})$, $\ldots\, \bar{q}^\sigma(a_{i+1}/a_i)$ become so small that

$$E(\bar{q}) - E_a(\bar{q}) \leq \big(E(\bar{q}^\varrho) - E_a(\bar{q}^\varrho)\big) \tag{B.23}$$

for all $\bar{q}$, which occur in an a-annihilation scheme leading from $\bar{q}_i^\sigma$ to $(m_i/m(\mathfrak{a}_{j(i)}))$ $\bar{q}^\sigma(\mathfrak{a}_{j(i)})$.

Therefore, the $\bar{q}^\lambda$ $(\lambda \geq \varkappa\,(N+1-p))$ are determined by a dissipation scheme, interrupted as often as possible by some a_i-annihilation scheme. If $V^{\varrho+1} R_0 \ldots V^\sigma$ is sufficiently connected, then one can reach $\bar{q}^\lambda = 0$ from the left. By proceeding from the right in a similar way one can find a solution $\bar{q}^\varkappa$ satisfying (1) and (4) in Lemma 4.2.

Let us check property (2). If $\bar{q}^\lambda = 0$ and $\tilde{q}^\lambda \in \mathsf{R}_0^{3N}$, then $E + i\,\varepsilon - E\,(\tilde{q}^\lambda\,(1 - i\,\varphi))$ never vanishes for $0 \leq \varphi \leq \varphi_0$ and $0 < \varepsilon$, and for $\varepsilon = 0$ and $0 < \varphi \leq \varphi_0$ only for $E = 0$, $\tilde{q}^\lambda = 0$. If (a) is satisfied and if $1 \leq \lambda < \varkappa\,(N-1)$, where $V^1 R_0 \ldots V^\lambda$ has connectivity a_i, then we use

$$E + i\,\varepsilon - E\,(\bar{q}^\lambda + \tilde{q}^\lambda\,(1 - i\,\varphi)) = E + i\,\varepsilon - E_{a_i}(k)$$

$$- \sum_{s=1}^{N} n_s \left[\bar{q}_s^\lambda - \frac{m_s}{m(\mathfrak{a}_{j(s)})}\,\bar{k}(\mathfrak{a}_{j(s)}) + \tilde{q}_s^\lambda\,(1 - i\,\varphi)\right]^2, \tag{B.24}$$

$$E - E_{a_i}(k) = E - E_{a_2}(k) + \eta_2\,k(a_2/a_1)^2$$

$$- \eta_2\,k(a_2/a_1)^2 - \sum_{j=3}^{i} n(a_j/a_{j-1})\,k(a_j/a_{j-1})^2 \tag{B.25}$$

and the fact that $\bar{q}_s^\lambda - (m_s/m(\mathfrak{a}_{j(s)}))\,\bar{k}(\mathfrak{a}_{j(s)})$ is a linear combination of $k(a_i/a_{i-1}), \ldots$ $k(a_2/a_1)$ with uniformly bounded coefficients. Therefore (B.24) never vanishes for $\varepsilon > 0$ and $0 \leq \varphi \leq \varphi_0$, $\varphi_0 > 0$ sufficiently small uniformly for all G, and for $\varepsilon \downarrow 0$ and $0 < \varphi < \varphi_0$ only if $E = 0$, $k(a_i/a_{i-1}) = \ldots = k(a_2/a_1) = 0$.

Assume that (b) holds. For the $\lambda \geq \varkappa\,(N+1-p)$, $\varphi > 0$, we use the identity

$$D^\lambda = E + i\,\varepsilon - E\,(\bar{q}^\lambda + \tilde{q}^\lambda\,(1 - i\,\varphi)) = E - E(\bar{q}^\lambda)$$

$$+ E(\tilde{q}^\lambda)\,(1 + \varphi^2) + \frac{\varepsilon}{\varphi} - \frac{Im\,D^\lambda}{\varphi} + i\,\mathrm{Im}\,D^\lambda \tag{B.26}$$

and $E(\tilde{q}^\lambda) \leq E(\tilde{q}^{\varkappa(N+1-p)})$, with equality only for $k(a_{p-1}/a_{p-2}) = \ldots = k(a_{i+1}/a_i) = 0$. ($a_i$: connectivity of $V^1 R_0 \ldots V^\lambda$). By construction

$$E - E(\bar{q}^\lambda) \geq E - E_{a_p}(k) + \frac{(n_g + h_h)^2}{n(\mathfrak{a}_{j(g)}) + n(\mathfrak{a}_{j(h)})}\,k(a_p/a_{p-1})^2$$

$$g = i\,\big(\varkappa(N+1-p)\big), \qquad h = j\,\big(\varkappa(N+1-p)\big) \tag{B.27}$$

Let $\eta = 1/2\,\min\,(n_g + n_h)^2/(n(\mathfrak{a}_{j(g)}) + n(\mathfrak{a}_{j(h)}))$. Then $D^\lambda \neq 0$ for $\varepsilon \geq 0$, if (B.19) is satisfied.

The other non-trivial configuration is $\lambda < \varkappa\,(N-p)$, with connectivity $a_r = (\mathfrak{a}_1', \ldots \mathfrak{a}_r')$ of $V^1 R_0 \ldots V^\lambda$.

Then

$$D^\lambda = E + i\,\varepsilon - E_{a_r}(k) - \sum_{s=1}^{N} n_s \left(\bar{q}_s^\lambda - \frac{m_s}{m(\mathfrak{a}_{j(s)}')}\,\bar{k}(\mathfrak{a}_{j(s)}') + \tilde{q}_s^\lambda\,(1 - i\,\varphi)\right)^2. \tag{B.28}$$

By construction, the $\bar{q}_s^\lambda - (m_s/m(\mathfrak{a}'_{j(s)}))\, \bar{k}(\mathfrak{a}'_{j(s)})$ are linear combinations of the relative momenta $k(a_r/a_{r-1}), \ldots k(a_{p+1}/a_p)$ with uniformly bounded coefficients. Furthermore

$$E - E_{a_r}(k) = E - E_{a_{p+1}}(k) + \eta_{p+1}\, k(a_{p+1}/a_p)^2$$

$$- \eta_{p+1}\, k(a_{p+1}/a_p)^2 - \sum_{j=p+2}^{r} n(a_j/a_{j-1})\, k(a_j/a_{j-1})^2 \tag{B.29}$$

and therefore (B.28) can vanish only for $k(a_r/a_{r-1}) = \ldots = k(a_{p+1}/a_p) = 0$ and $E = E_{a_{p+1}}(k)$, if φ_0 is sufficiently small.

The finiteness (4) of different classes of solutions follows from the interposition of a_i-annihilation schemes, whenever this is possible. The HC finally is a consequence of the fact that the different determinations (B.17) or (B.19) become equal with a power law at the thresholds.

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