On perturbation expansions in axiomatic quantum field theory

Autor(en): **Curtius, P.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **36 (1963)**

Heft IV

PDF erstellt am: **27.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-113379

On Perturbation Expansions in Axiomatic Quantum Field Theory

by **P. Curtius**
Theoretisch-Physikalisches Seminar der E.T.H., Zürich

(1. XI. 62)

Abstract: It is shown that the perturbation expansions of L. S. Z. axiomatic field theory can be described by Lagrangian formalism; and conversely that the perturbation series derived from Lagrangian formalism satisfy the conditions following from a perturbation treatment of the L. S. Z. axioms.

1. Introduction

In this paper the connection between the perturbation treatments of axiomatic and of Lagrangian quantised field theories is examined. In sections 2 and 3, the framework of the L. S. Z. axioms and the method of generating functionals are developed. In sections 4 and 5, equivalence theorems connecting the axioms and properties of certain vacuum expectation values are derived for the complete fields and for their perturbation expansions. In section 6, the functional representation of the unrenormalised Feynman-Dyson series is shown to formally satisfy the τ-equations in every order. In section 7, the same result is derived after renormalisation has been carried out, and it is shown conversely that all solutions of the time-ordered τ-equations can be described by a renormalised Feynman-Dyson series. Section 8 contains consequences of this result following from the equivalence theorems. The proofs are complete if the integrability conditions (3), (3′) defined in section 4 are satisfied in all orders of perturbation theory.

2. The L.S.Z. Axioms

The discussion in this paper is confined to the theory of a single Hermitian scalar field with non-vanishing rest mass, which is described in the Heisenberg picture by the symmetric field operator $A(x)$. For this theory, the axiomatic framework developed in the papers of LEHMANN, SYMANZIK, and ZIMMERMANN[1]) is used; we quote the L. S. Z. axioms in the following form:

1. (*Operator distribution*): In a Hilbert space $\mathfrak{H}$ with positive definite scalar product (Φ, Ψ), a symmetric temperate operator distribution $A(x)$ shall be defined on a dense subset $\mathcal{D} \subset \mathfrak{H}$, further restrictions on $\mathcal{D}$ to be made subsequently. For such solutions

$f^\alpha(x)$ of the Klein-Gordon equation for mass m which are "testing functions with fast decrease" (see L. SCHWARTZ[2]) on the mass shell, the expressions

$$A^\alpha(t)\,\Phi \qquad \Phi \in \mathcal{D}$$

shall exist*), where

$$A^\alpha(t) \equiv i \int_{x_0=t} A(x)\,\tilde{\partial}_0 f^\alpha(x)\,d^3x\,; \qquad F\,\tilde{\partial}_0 G \equiv F\,\dot{G} - \dot{F}\,G$$

and $A^\alpha(t)\;\mathcal{D} \subset \mathcal{D}$.

2. (*Lorentz Invariance*): In $\mathfrak{H}$ there shall exist a continuous unitary representation $U(a, \Lambda)$ of the real inhomogeneous proper orthochronous Lorentz group $L_+^\uparrow$ with

$$U(a, \Lambda)\;\mathcal{D} \subset \mathcal{D}\,; \qquad U(a, \Lambda)\,A(x)\,U^{-1}(a, \Lambda) = A\,(\Lambda\,x + a)\,. \tag{2.1}$$

There shall exist one (and only one up to multiplication by c-number factors) vector $\Omega \in \mathcal{D}$ which is invariant against all transformations $U(a, \Lambda)$, $(a, \Lambda) \in L_+^\uparrow$.

3. (*Locality*): For $x - y$ space-like,

$$(\Phi, [A(x), A(y)]\,\Psi) = 0 \qquad \Phi, \Psi \in \mathcal{D}\,. \tag{2.2}$$

4. (*Asymptotic Condition*): There shall exist in $\mathfrak{H}$ free fields $A_{in}(x)$, $A_{out}(x)$ of mass m defined on $\mathcal{D}$ such that $A^\alpha_{\substack{in\\out}}(t)\;\mathcal{D} \subset \mathcal{D}$, and

$$\lim_{t \to \substack{-\infty\\+\infty}} (\Phi, A^\alpha(t)\,\Psi) = \left(\Phi, A^\alpha_{\substack{in\\out}}(t)\,\Psi\right) \qquad \Phi, \Psi \in \mathcal{D}\,. \tag{2.3}$$

For a complete system of f^α, Ω shall be cyclic with respect to the ring of polynomials either in A^α_{in} or A^α_{out}.

From these assumptions follows the existence of a unitary scattering matrix S defined by

$$A_{out}(x) = S^{-1}\,A_{in}(x)\,S\,. \tag{2.4}$$

It also follows that $\mathcal{D}$ contains a complete orthonormal system of vectors constructed by application of polynomials either in A^α_{in} or in A^α_{out} on Ω.

For matrix elements between vectors from $\mathcal{D}$, the asymptotic condition further permits to derive reduction formulae

$$\left.\begin{aligned} &[A_{in}(x), S\,T[A(x_1)\,\ldots\,A(x_n)]] = \\ &\qquad -\int \Delta(x - x')\,K_{x'}\,S\,T\,[A(x')\,A(x_1)\,\ldots\,A(x_n)]\,dx'\,, \end{aligned}\right\} \tag{2.5a}$$

$$\left.\begin{aligned} &[A_{in}(x), R[A(y); A(x_1)\,\ldots\,A(x_n)]] \\ &= i\int \Delta(x - x')\,K_{x'}\,R\,[A(y); A(x')\,A(x_1)\,\ldots\,A(x_n)]\,dx'\,; \quad K_x \equiv -\Box + m^2 \end{aligned}\right\} \tag{2.5b}$$

*) O. W. GREENBERG (Princeton Thesis 1956) has shown that in the case of diverging wave function renormalisation, this condition has to be weakened; one then only postulates existence of the convolution product of these expressions with a testing function in the time variable.

for the T- and R-products of the field operator defined*) by

$$T[A(x_1) \ldots A(x_n)] \equiv \sum_{P(1 \ldots n)} \theta(x_1 - x_2) \ldots \theta(x_{n-1} - x_n) \, A(x_1) \ldots A(x_n) \,, \tag{2.6a}$$

$$\left.\begin{aligned} &R[A(x); A(x_1) \ldots A(x_n)] \\ &\quad \equiv \sum_{P(1 \ldots n)} i^n \, \theta(x - x_1) \ldots \theta(x_{n-1} - x_n) \, [[\ldots [A(x), A(x_1)], \ldots], A(x_n)] \,. \end{aligned}\right\} \tag{2.6b}$$

The summation extends over all permutations of the indices $1, \ldots, n$.

These reduction formulae permit expression of the matrix elements of T- and R-products between vectors of the "incoming" orthonormal system by vacuum expectation values ("v.e.v") of T- and R-products of higher degree. The result of the reduction procedure can be collected in the following formal series expansion (valid only for matrix elements between vectors from $\mathcal{D}$):

$$\left.\begin{aligned} &S \, T[A(x_1) \ldots A(x_n)] \\ &\quad = \sum_{\nu=0}^{\infty} \frac{i^\nu}{\nu!} \int \ldots \int \tau(x_1 \ldots x_n \, \bar{u}_1 \ldots \bar{u}_\nu) : A_{in}(u_1) \ldots A_{in}(u_\nu) : du_1 \ldots du_\nu \,, \end{aligned}\right\} \tag{2.7a}$$

$$\left.\begin{aligned} &R[A(y); A(x_1) \ldots A(x_n)] \\ &\quad = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \int \ldots \int r(y; x_1 \ldots x_n \, \bar{u}_1 \ldots \bar{u}_\nu) : A_{in}(u_1) \ldots A_{in}(u_\nu) : du_1 \ldots du_\nu \,. \end{aligned}\right\} \tag{2.7b}$$

For $n = 0$, these formulae reduce to series representations of S and $A(x)$. The τ- and r-functions are defined by

$$\tau(x_1 \ldots x_n) \equiv (\Omega, \, T[A(x_1) \ldots A(x_n)] \, \Omega) \,, \tag{2.8a}$$

$$r(y; x_1 \ldots x_n) \equiv (\Omega, \, R[A(y); A(x_1) \ldots A(x_n)] \, \Omega) \tag{2.8b}$$

where barred coordinates designate "amputation":

$$\tau(\ldots \bar{x}_\nu \ldots) = i \, K_{x_\nu} \, \tau(\ldots x_\nu \ldots) \,, \tag{2.9a}$$

$$r(x; \ldots \bar{x}_\nu \ldots) = K_{x_\nu} \, r(x; \ldots x_\nu \ldots) \,, \tag{2.9b}$$

$$r(\bar{x}; \ldots) = K_x \, r(x; \ldots) \,. \tag{2.9c}$$

The inversion of the amputation prescription is

$$\tau(\ldots x_\nu \ldots) = \int \Delta^F(x_\nu - y_\nu) \, \tau(\ldots \bar{y}_\nu \ldots) \, dy_\nu \,, \tag{2.10a}$$

$$r(x; \ldots x_\nu \ldots) = \int \Delta^A(x_\nu - y_\nu) \, r(x; \ldots \bar{y}_\nu \ldots) \, dy_\nu \,, \tag{2.10b}$$

$$r(x; \ldots) = \int \Delta^R(x - y) \, r(\bar{y}; \ldots) \, dy \,. \tag{2.10c}$$

---

*) For the definition of products of temperate distributions with θ-functions see here O. STEINMANN[3]).

3. Formal Generating Functionals

The algebraic properties of the infinite systems of T- and R-products are very concisely expressed with the aid of formal generating functionals[4]). A generating functional $\mathcal{F}\{J\}$ of a function variable $J(x)$

$$\mathcal{F}\{J\} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \ldots \int f^{(n)}(x_1 \ldots x_n)\, J(x_1) \ldots J(x_n)\, dx_1 \ldots dx_n \tag{3.1}$$

may be considered as a Volterra expansion (analogous to TAYLOR's series)

$$\mathcal{F}\{I + J\} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int J(x) \frac{\delta}{\delta_{I(x)}}\, dx\right]^n \cdot \mathcal{F}\{I\} \equiv e^{J \frac{\delta}{\delta I}} \cdot \mathcal{F}\{I\} \tag{3.2}$$

of $\mathcal{F}$ at $I = 0$. The Volterra coefficients $f^{(n)}$ can be considered as formal functional derivatives defined by

$$\left[\frac{\delta}{\delta_{J(x)}}, J(x')\right] = \delta^4(x - x') \tag{3.3}$$

$f^{(n)}(x_1 \ldots x_n)$ is symmetric in $x_1, \ldots, x_n$.

We expressly state that in this paper we shall not need any assumptions concerning the convergence of Volterra series, or concerning the existence of generating functionals or their functional derivatives as operators in $\mathfrak{H}$; generating functionals are used exclusively as a means of writing operations defined on each of their Volterra coefficients, in analogy to the calculus of formal power series.

The generating functionals of T- and R-products are defined by

$$\left.\begin{aligned}\mathcal{T}\{J\} = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \ldots \int T[A(x_1) \ldots A(x_n)]\, J(x_1) \ldots J(x_n)\, dx_1 \ldots dx_n \\ \equiv T \exp\left\{i \int A(x)\, J(x)\, dx\right\},\end{aligned}\right\} \tag{3.4a}$$

$$\left.\begin{aligned}\mathfrak{R}_x\{J\} = A(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \ldots \int R[A(x); A(x_1) \ldots A(x_n)] \\ \times J(x_1) \ldots J(x_n)\, dx_1 \ldots dx_n .\end{aligned}\right\} \tag{3.4b}$$

Some of their formal properties are contained in the following lemmas (for the proof one may e.g. refer to the author's dissertation[6])):

Lemma 1 (*Time-ordered Functionals*): If $\mathcal{S}\{J\}$ is the formal generating functional of an infinite system of operator-valued distributions $\{S[x_1 \ldots x_n]\}$

$$\mathcal{S}\{J\} = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \ldots \int S[x_1 \ldots x_n]\, J(x_1) \ldots J(x_n)\, dx_1 \ldots dx_n \tag{3.5}$$

then the relation

$[\bar{O}]$*) $$S[x_1 \ldots x_n] = S\; T[A(x_1) \ldots A(x_n)] \tag{3.6}$$

*) The bracket $[\bar{O}]$ is to signify that an equation is valid up to an uncertainty with support in the subspace $O = [x_1, \ldots, x_n; x_1 = \ldots = x_n]$; $x_1, \ldots, x_n$ are all the variables explicitly contained in the equation.

is necessary and sufficient for the equations

$$[\bar{O}] \qquad \theta(y-x)\,\frac{\delta}{\delta_{J(y)}}\,\mathcal{S}^*\{J\}\,\mathcal{S}_{,x}\{J\}=0 \qquad \mathcal{S}_{,x}\equiv\frac{\delta}{\delta_{J(x)}}\,\mathcal{S}. \tag{3.7a}$$

$$A(x)=\frac{1}{i}\,\mathcal{S}^*\{J\}\,\mathcal{S}_{,x}\{J\}\Big|_{J=0} \tag{3.7b}$$

S is a unitary operator independent of J. [(3.7a) will be referred to as BOGOLJUBOV's equation[7]); it implies unitarity as well as time-ordering of $\mathcal{S}$, if products of temperate distributions with θ-functions are defined such that $\theta(x)+\theta(-x)=1$ is preserved.]

Lemma 2 (*Retarded Functionals*): If $Q_x\{J\}$ is the formal generating functional of an infinite system of operator-valued distributions $\{Q[x;x_1\ldots x_n]\}$,

$$Q_x\{J\}=Q[x]+\sum_{n=1}^{\infty}\frac{1}{n!}\int\cdots\int Q[x;x_1\ldots x_n]\,J(x_1)\ldots J(x_n)\,dx_1\ldots dx_n \tag{3.8}$$

then the relation

$$[\bar{O}] \qquad Q[x;x_1\ldots x_n]=R[A(x);A(x_1)\ldots A(x_n)] \tag{3.9}$$

is necessary and sufficient for the equations

$$[\bar{O}] \qquad Q_{x,y}\{J\}=i\,\theta(x-y)\,[Q_x\{J\},Q_y\{J\}]\,, \tag{3.10a}$$

$$Q[x]=A(x)\,. \tag{3.10b}$$

[(3.10a) will be referred to as Peierls' equation[8]); it implies the retardation of $\mathfrak{R}_x$ as well as the "unitarity" equation $\mathfrak{R}_{x,y}-\mathfrak{R}_{y,x}=i\,[\mathfrak{R}_x,\mathfrak{R}_y]$.]

4. Equivalence Theorems

The operator-valued generating functionals of the preceding section can be expressed by their v.e.v.

$$\mathfrak{t}\{J\}=1+\sum_{n=2}^{\infty}\frac{i^n}{n!}\int\cdots\int\tau(x_1\ldots x_n)\,J(x_1)\ldots J(x_n)\,dx_1\ldots dx_n\,, \tag{4.1a}$$

$$\mathfrak{r}_x\{J\}=\sum_{n=1}^{\infty}\frac{1}{n!}\int\cdots\int r(x;x_1\ldots x_n)\,J(x_1)\ldots J(x_n)\,dx_1\ldots dx_n \tag{4.1b}$$

with the aid of the reduction formulae. The expansions (2.7) are equivalent to the following expression for the generating functionals:

$$S\cdot\mathcal{T}\{J\}=:\mathfrak{t}\,\{J+A_{in}K_{x'}\}:\equiv:\exp\left\{\int A_{in}(x')\,K_{x'}\,\frac{\delta}{\delta_{J(x')}}\,dx'\right\}:\mathfrak{t}\{J\}\,, \tag{4.2a}$$

$$\mathfrak{R}_x\{J\}=:\mathfrak{r}_x\,\{J+A_{in}K_{x'}\}:\equiv:\exp\left\{\int A_{in}(x')\,K_{x'}\,\frac{\delta}{\delta_{J(x')}}\,dx'\right\}:\mathfrak{r}_x\{J\}\,. \tag{4.2b}$$

These expressions are valid for matrix elements between vectors from $\mathcal{D}$; since $\mathcal{D}$ contains a complete system of orthonormal state vectors constructed by application of polynomials in A^{α}_{in} on Ω, matrix elements of non-linear expressions in the functionals (4.2) can be expressed by summation over this system of intermediate states. It is well-known that the summation over states yields expressions in the v. e. v. of the form

$$\mathfrak{R}_x\{J\}\,\mathfrak{R}_y\{J\} = :\exp\left\{\int A_{in}(x')\,K_{x'}\,\frac{\delta}{\delta_{J(x')}}\,dx'\right\}: \mathfrak{r}_x\{J\} * \Delta^+ * \mathfrak{r}_y\{J\}, \qquad (4.3a)$$

where the "Δ^+-convolution" of the two v. e. v.-functionals is defined by

$$\mathfrak{r}_x\{J\} * \Delta^+ * \mathfrak{r}_y\{J\} = \mathfrak{r}_x\{J\} \cdot \exp\left\{\iint \overleftarrow{\frac{\delta}{\delta_{J(u)}}\,K_u}\; i\,\Delta^+(u-v)\; \overrightarrow{K_v\,\frac{\delta}{\delta_{J(v)}}}\,du\,dv\right\} \mathfrak{r}_y\{J\}. \qquad (4.3b)$$

The Δ^+-convolution can be formally derived from the product of normal products (4.2b) via

$$e^a\,e^b = e^b\,e^a\,e^{[a,b]}, \quad \text{if } [a,[a,b]] = [b,[a,b]] = 0. \qquad (4.4)$$

In any field theory with L. S. Z. asymptotic condition, it is therefore possible to derive an infinite system of non-linear integral equations for the τ-functions and for the r-functions by insertion of the expansions (4.2) in the Bogoljubov equation (3.7a) and in the Peierls equation (3.10a). The Bogoljubov equation for τ-functions[9] is

$$\left.\begin{aligned}
&\theta\,(y-x)\sum_{k=0}^{n} P^k(-1)^k \sum_{l=0}^{\infty}\int\cdots\int\prod_{\nu=1}^{l}[du_\nu\,dv_\nu\,i\,\Delta^+(u_\nu-v_\nu)]\\
&\times\{[\tau^*(x_1\ldots x_k\,\bar{u}_1\ldots\bar{u}_l)\,\tau(x\,y\,x_{k+1}\ldots x_n\,\bar{v}_1\ldots\bar{v}_l)-\tau^*(y\,x_1\ldots x_k\,\bar{u}_1\ldots\bar{u}_l)\\
&\times\tau(x\,x_{k+1}\ldots x_n\,\bar{v}_1\ldots\bar{v}_l)]\}+\frac{i}{2}\,\sigma(x,y,x_1,\ldots,x_n)=0
\end{aligned}\right\}\qquad(4.5a)$$

while the Peierls equation for r-functions is [10][11]

$$\left.\begin{aligned}
r(x;y\,x_1\ldots x_n) = {}& i\,\theta\,(x-y)\sum_{k=0}^{n}P^k\sum_{l=1}^{\infty}\int\cdots\int\prod_{\nu=1}^{l}[du_\nu\,dv_\nu\,i\,\Delta^+(u_\nu-v_\nu)]\\
&\times\{r(x;x_1\ldots x_k\,\bar{u}_1\ldots\bar{u}_l)\,r(y;x_{k+1}\ldots x_n\,\bar{v}_1\ldots\bar{v}_l)-[x\leftrightarrow y]\}\\
&+\varrho(x;y,x_1\ldots x_n).
\end{aligned}\right\}\qquad(4.5b)$$

The prescription $P^k f(x_1\ldots x_k)\,g(x_{k+1}\ldots x_n)$ is defined as summation over all divisions of the arguments $x_1,\ldots x_n$ into two classes of k and $n-k$ elements respectively.

The uncertainties appearing in the functional equations of the lemmas 1 and 2 are here exhibited through undetermined real distributions $\sigma(x,y,x_1,\ldots,x_n)$ and $\varrho(x;y,x_1,\ldots,x_n)$ with support in $\{x=y=x_1=\ldots=x_n\}$; their reality is implied by the unitarity property contained in (3.7a) and (3.10a). In these equations, we have set the unimodular constant $\mathfrak{t}\{0\}=1$.

If, on the other hand, Lorentz-invariant solutions $\{\tau\}$ or $\{r\}$ of (4.5) are given, then (with the aid of a free field $A_{in}(x)$ with cyclic vacuum Ω) one can construct by the series (4.2) a field theory satisfying the L. S. Z. axioms, if the following convolution integrals exist as temperate distributions:

$$\int \cdots \int \tau^*(\bar{x}_1 \ldots \bar{x}_m)\, \tau(\bar{y}_1 \ldots \bar{y}_n) \cdot \prod_{\{\lambda_i\} \subset \{1,2,\ldots Min\,(m,n)\}} [\Delta^+(x_{\lambda_i} - y_{\lambda_i})\, dx_{\lambda_i}\, dy_{\lambda_i}]\,, \tag{4.6a}$$

$$\int \cdots \int r(\bar{x}; \bar{x}_1 \ldots \bar{x}_m)\, r(\bar{y}; \bar{y}_1 \ldots \bar{y}_n) \cdot \prod_{\{\lambda_i\} \subset \{1,2,\ldots Min\,(m,n)\}} [\Delta^+(x_{\lambda_i} - y_{\lambda_i})\, dx_{\lambda_i}\, dy_{\lambda_i}]\,. \tag{4.6b}$$

Locality is here a consequence of Lorentz invariance and time-ordering or retardation.

From lemma 2 follows in this way the *G.L.Z.-Theorem:* If the symmetric field operator $A(x)$ satisfies the L. S. Z. axioms, then the r-functions defined from it through (2.8b) satisfy the following conditions:

1. $r(x; x_1 \ldots x_n)$ is a real Lorentz-invariant temperate distribution symmetric in $x_1, \ldots, x_n$.
2. The r-functions satisfy the Peierls-equation (4.5b).
3. The convolution integrals (4.6b) exist as temperate distributions. –

Given on the other hand a set of r-functions satisfying conditions 1, 2, and 3, then

$$A(x) = :\mathfrak{r}_x \{A_{in}\, K_{x'}\}: \tag{4.7a}$$

is a representation (valid only for matrix elements between vectors from $\mathscr{D}$) of a symmetric field operator $A(x)$ satisfying the L. S. Z. axioms, the R-products of which fulfil equation (2.8b). –

This theorem has been proved by GLASER, LEHMANN, and ZIMMERMANN[12]). From lemma 1 follows in the same way the

"Time-Ordered G.L.Z.-Theorem": If the symmetric field operator $A(x)$ satisfies the L. S. Z. axioms, then the τ-functions defined from it through (2.8a) fulfil the following conditions:

1′. $\tau(x_1 \ldots x_n)$ is a Lorentz-invariant temperate distribution symmetric in $x_1, \ldots, x_n$.
2′. The τ-functions satisfy the Bogoljubov-equation (4.5a).
3′. The convolution integrals (4.6a) exist as temperate distributions. –

Given on the other hand a set of τ-functions satisfying these conditions, then

$$A(x) = \frac{1}{i} : \exp\left\{\int A_{in}(y')\, K_{y'} \frac{\delta}{\delta_{J(x')}}\, dx'\right\} : \mathfrak{t}^*\{J\} * \Delta^+ * \mathfrak{t}_{,x}\{J\}\,, \tag{4.7b}$$

where A_{in} is a free field with cyclic vacuum vector Ω, is a representation (valid for matrix elements between vectors from $\mathscr{D}$) of a symmetric field operator $A(x)$ satisfying the L. S. Z. axioms, the T-products of which fulfil equation (2.8a). –

By combination of the two G. L. Z.-theorems follows immediately the *"τ-r-Equivalence Theorem":* Given a set of τ-functions satisfying the conditions 1′, 2′, and 3′; then the r-functions defined by

$$\mathfrak{r}_x\{J\} = \frac{1}{i}\, \mathfrak{t}^*\{J\} * \Delta^+ * \mathfrak{t}_{,x}\{J\} \tag{4.8}$$

satisfy the conditions 1, 2, and 3. Given on the other hand a set of r-functions satisfying these conditions, then the τ-functions defined by

$$\tau(x_1 \ldots x_n) = T\,[\mathfrak{r}_{x_1} * \Delta^+ * (\mathfrak{r}_{x_2} * \Delta^+ * (\ldots \mathfrak{r}_{x_n}) \ldots)]\,|_{J=0} \tag{4.9}$$

satisfy the conditions 1′, 2′, and 3′. –
A proof of the last two theorems has been given in the author's dissertation.

5. Perturbation Expansions

Having in mind the well-known perturbation treatment of Lagrangian field theory to be considered below, we define perturbation expansions in terms of a coupling constant g of the field operator

$$A(x) = \sum_{m=0}^{\infty} g^m A^{(m)}(x) \tag{5.1}$$

and of the Green's functions

$$\tau(x_1 \ldots x_n) = \sum_{m=0}^{\infty} g^m \tau^{(m)}(x_1 \ldots x_n)\,, \tag{5.2a}$$

$$r(x; x_1 \ldots x_n) = \sum_{m=0}^{\infty} g^m r^{(m)}(x; x_1 \ldots x_n)\,. \tag{5.2b}$$

As usual, these expansions are treated as formal power series without assumptions concerning their convergence. The aim of this section is to show in what way the perturbation treatment modifies the equivalence theorems.

We define perturbation theoretic L. S. Z. axioms (1p), (2p), and (4p) as the postulate of validity of the axioms 1, 2, and 4 (with $A^{(0)} = A_{in}$) for every order of the partial sums

$$A_n(x) = \sum_{m=0}^{n} g^m A^{(m)}(x)\,. \tag{5.3}$$

In order to simplify the treatment of operator domains, we confine the discussion of this paper to the case that the expressions (4.7) of the field operator are of finite degree in $A^{(0)}(x)$ in every order of perturbation theory (this corresponds to the case of Lagrangians of finite degree in the interacting field operator). The domain $\mathcal{D}$ of the L. S. Z. axioms is then in all instances to be substituted by the domain $\mathcal{D}^0$ of the free field operator $A^{(0)}(x)$. Thus, perturbation theoretic locality [axiom (3p)] is defined by ($x - y$ spacelike)

$$\left(\Phi, \sum_{k=0}^{m} [A^{(k)}(x), A^{(m-k)}(y)]\,\Psi\right) = 0\,, \quad \Phi, \Psi \in \mathcal{D}^0\,, \quad m = 0, 1, 2, \ldots \tag{5.4}$$

We further define G. L. Z.-conditions (1p), (3p), and (1′p), (3′p) to mean the validity of conditions 1, 3, and 1′, 3′ respectively for all $r^{(m)}$ of $\tau^{(m)}$ in every perturbation order m.

Conditions (2p) and (2′p) are defined as the validity of the perturbation theoretic Peierls and Bogoljubov equations:

$$\left.\begin{aligned} r^{(m)}(\bar{x};\bar{y}\,\bar{x}_1\ldots\bar{x}_n) &= i\,\theta\,(x-y)\sum_{j=1}^{m-1}\sum_{k=0}^{n}P^k\sum_{l=1}^{\infty}\int\cdots\int\prod_{\varrho=1}^{l}[du_\varrho\,dv_\varrho\,i\,\Delta^+(u_\varrho-v_\varrho)] \\ &\quad\times\{r^{(j)}(\bar{x};\bar{x}_1\ldots\bar{x}_k\,\bar{u}_1\ldots\bar{u}_l)\,r^{(m-j)}(\bar{y};\bar{x}_{k+1}\ldots\bar{x}_n\,\bar{v}_1\ldots\bar{v}_l)-[x\leftrightarrow y]\} \\ &\quad+\varrho^{(m)}(x;y,x_1,\ldots,x_n)\,. \end{aligned}\right\}\quad(5.5)$$

$$\left.\begin{aligned} \tau^{(m)}(\bar{x}\,\bar{y}\,\bar{x}_1\ldots\bar{x}_n) &= -\,\theta\,(y-x)\Bigg\{\sum_{j=0}^{m}\sum_{k=1}^{n-1}P^k\,[\tau^{(j)*}(\bar{y}\,\bar{x}_1\ldots\bar{x}_k)\,\tau^{(m-j)}\,(\bar{x}\,\bar{x}_{k+1}\ldots x_n) \\ &\quad+\tau^{(j)*}(\bar{x}_1\ldots\bar{x}_k)\,\tau^{(m-j)}(\bar{y}\,\bar{x}\,\bar{x}_{k+1}\ldots\bar{x}_n) \\ &\quad+\sum_{j=1}^{m-1}\sum_{k=0}^{n}P^k\sum_{l=1}^{\infty}\int\cdots\int\prod_{\varrho=1}^{l}[du_\varrho\,dv_\varrho\,i\,\Delta^+(u_\varrho-v_\varrho)] \\ &\quad\times[\tau^{(j)*}(\bar{y}\,\bar{x}_1\ldots\bar{x}_k\,\bar{u}_1\ldots\bar{u}_l)\,\tau^{(m-j)}(\bar{x}\,\bar{x}_{k+1}\ldots\bar{x}_n\,\bar{v}_1\ldots\bar{v}_l) \\ &\quad+\tau^{(j)*}(\bar{x}_1\ldots\bar{x}_k\,\bar{u}_1\ldots\bar{u}_l)\,\tau^{(m-j)}(\bar{y}\,\bar{x}\,\bar{x}_{k+1}\ldots\bar{x}_n\,\bar{v}_1\ldots\bar{v}_l)]\Bigg\} \\ &\quad-[x\leftrightarrow y]+i\,\sigma^{(m)}(x,y,x_1,\ldots,x_n)\,. \end{aligned}\right\}\quad(5.6)$$

$\varrho^{(m)}(x;x_1\ldots x_n)$ and $\sigma^{(m)}(x_1\ldots x_n)$ are real, Lorentz-invariant temperate distributions symmetric in $x_1,\ldots,x_n$ with support in $x=x_1=\ldots=x_n$.

We now insert the series (5.1), (5.2) in the proofs of the first and third equivalence theorems of the preceding section. The calculus of formal power series then yields the following perturbation theoretic lemmas, which are to be applied below:

Lemma 3: Given a perturbation expansion (5.1) of the symmetric field operator $A(x)$ which satisfies axioms (1p) through (4p). Then the perturbation theoretic r-functions defined by

$$r^{(m)}(x;x_1\ldots x_n)=\sum_{m_0+\cdots+m_n=m}(\Omega,\,R[A^{(m_0)}(x);A^{(m_1)}(x_1)\ldots A^{(m_n)}(x_n)]\,\Omega)\quad(5.7)$$

satisfy the perturbation theoretic G. L. Z.-conditions (1p), (2p), and (3p). – Given on the other hand a perturbation expansion (5.2b) of the r-functions satisfying these conditions*); then the series (4.7a) constructed from the perturbation sums of the $r^{(m)}$ with the aid of a free field A^0 with cyclic vacuum vector is a representation of a symmetric field operator (valid for matrix elements between vectors from $\mathcal{D}^0$), which satisfies the perturbation theoretic L. S. Z. axioms and equation (5.7).

Lemma 4: Given a perturbation expansion (5.2a) of the τ-functions satisfying the conditions (1′p), (2′p), and (3′p); then the $r^{(m)}$-functions defined by

$$\mathfrak{r}_x^{(m)}\{J\}=\frac{1}{i}\sum_{k=0}^{m}\mathfrak{t}^{(k)*}\{J\}*\Delta^+*\mathfrak{t}_{,x}^{(m-k)}\{J\}\quad(5.8)$$

*) For the question of symmetry of the solutions of (5.5) cf. H. M. Fried[13]).

satisfy the conditions (1p), (2p), and (3p). – Given on the other hand a perturbation expansion (5.2b) of the r-functions satisfying these conditions, then the $\tau^{(m)}$-functions recursively defined by

$$\left.\begin{aligned}\tau^{(m)}(\bar{x}\,\bar{x}_1\ldots\bar{x}_n) &= i\sum_{j=0}^{m}\sum_{k=2}^{n-1} P^k\,\tau^{(j)}(\bar{x}_1\ldots\bar{x}_k)\,r^{(m-j)}(\bar{x};\bar{x}_{k+1}\ldots\bar{x}_n)\\ &\quad + i\sum_{j=1}^{m-1}\sum_{k=0}^{n} P^k\sum_{l=1}^{\infty}\int\ldots\int\prod_{\varrho=1}^{l}[du_\varrho\,dv_\varrho\,i\,\Delta^+(u_\varrho-v_\varrho)]\\ &\quad \times\tau^{(j)}(\bar{x}_1\ldots\bar{x}_k\,\bar{u}_1\ldots\bar{u}_l)\,r^{(m-j)}(\bar{x};\bar{x}_{k+1}\ldots\bar{x}_n\,\bar{v}_1\ldots\bar{v}_l)\end{aligned}\right\}\quad(5.9)$$

satisfy the conditions (1'p), (2'p), and (3'p). –

6. Unrenormalised Perturbation Theory of Lagrangian Formalism

By the equivalence theorems the investigation of the connection between "axiomatic" and Lagrangian perturbation theory may be reduced to verification of the conditions (1'p), (2'p), and (3'p) for the perturbation theoretic τ-functions derived from Lagrangian theory. In this section it will be shown that the unrenormalised Feynman integrals formally satisfy the perturbation theoretic Bogoljubov equation (5.6) in every order.

In the proof, the formal summation of FEYNMAN's perturbation series via generating functionals will be used. The easiest access to this method is by SCHWINGER's functional equation for the time-ordered functional $\mathcal{T}\{J\}$. Given a Lagrange theory with local Lorentz-invariant interaction term

$$L_w[A(x)] = \sum_n g\,L_n^{(1)}\,[A(x)] \qquad (6.1)$$

which for simplicity is assumed to be linear in a single coupling constant g. The $L_n^{(1)}$ shall be hermitian Lorentz-invariant polynomials homogeneous of n-th degree in the field operator $A(x)$ or its space-time derivatives of finite order. Then application of the equation of motion and of the canonical commutators to the formal equation

$$\mathcal{T}_{,x}\{I\} = T\left[\exp\left\{i\int_{x_0}^{+\infty} A(u)\,I(u)\,du\right\}\right] i\,A(x)\,T\left[\exp\left\{i\int_{-\infty}^{x_0} A(v)\,I(v)\,dv\right\}\right]$$

yields (SCHWINGER[14])

$$\left.\begin{aligned}&\left\{K_x\frac{1}{i}\frac{\delta}{\delta I(x)} - L'_w\left[\frac{1}{i}\frac{\delta}{\delta I(x)}\right] - I(x)\right\}\mathcal{T}\{I\} = 0\,;\\ &\qquad\qquad L'_w[f(x)] \equiv \frac{\delta}{\delta f(x)}\int L'_w\,[f(x')]\,dx'\,.\end{aligned}\right\}\quad(6.2)$$

Schwinger's equation is of course also valid for the v.e.v. of the $\mathcal{T}$-functional which generates the τ-functions. For $L_w = 0$, the generating functional of the free propagators

$$\mathfrak{t}^{(0)}\{I\} = \exp\left\{-\frac{1}{2}\iint I(u)\,\Delta^F\,(u-v)\,I(v)\,du\,dv\right\} \equiv e^{-1/2\,I\,\Delta^F I}\,, \qquad (6.3a)$$

$$\Delta^F(x) = i\,\theta(x)\,\Delta^+(x) + i\,\theta(-x)\,\Delta^+(-x)\,, \qquad (6.3b)$$

is a solution of (6.2). If $L_w \neq 0$,

$$\mathfrak{t}\{I\} = \exp\left\{i\int L_w\left[\frac{1}{i}\,\frac{\delta}{\delta\, I(x)}\right] dx\right\} e^{-1/2\, I\, \Delta^F I} \tag{6.4}$$

is a solution of (6.2), as may be seen through commutation of $I(x)$ with the first factor on the right. It is easily verified that the expansion in powers of the coupling constant of the τ-functions effected by the differentiation prescription (6.4) is after amputation identical term by term with the unrenormalised Feynman perturbation integrals[15]. These integrals are in general divergent, and all our manipulations of unrenormalised Feynman integrals are formal; i.e. multiplication and integration of distributions is effected regardless of existence questions. In these manipulations the following rules will be observed:

1. Graphs with "short-circuited" lines (beginning and ending on the same vertex) are disregarded.

2. If after insertion of the definition (3.6b) of Δ^F an integrand contains a product which by permutation of the order of distribution factors can be brought to the form

$$\theta\,(x_1 - x_2)\,\ldots\,\theta\,(x_{n-1} - x_n)\;\theta\,(x_n - x_1)\; F(x_{i_1}\,\ldots\, x_{i_\nu})\,;\qquad x_i \in \{x_1, \ldots, x_n\} \tag{6.5}$$

the corresponding integral is put equal to zero.

The perturbation theoretic Bogoljubov equation (5.6) is in functional notation

$$[\overline{O}]\qquad \theta\,(y - x)\,\frac{\delta}{\delta\, I(y)}\sum_{k=0}^{m} \mathfrak{t}^{(k)*}\{I\} * \Delta^+ * \mathfrak{t}_x^{(m-k)}\{I\} = 0\,; \tag{6.6}$$

the significance of the clause $[\overline{O}]$ has been explained in the footnote to equation (3.6). The verification of (6.6) can be reduced to a linear problem through SYMANZIK's generating functional of double graph integrals.

SYMANZIK's functional[16] is defined by

$$\Re\{\overline{J}, J\} = \mathcal{T}^*\left\{J + \frac{i}{2}\,\overline{J}\right\}\mathcal{T}\left\{J - \frac{i}{2}\,\overline{J}\right\}. \tag{6.7}$$

Its v.e.v. $\mathfrak{r}\,\{\overline{J}, J\}$ can be calculated from $\mathfrak{t}\{J\}$ by Δ^+-convolution:

$$\mathfrak{r}\{\overline{J}, J\} = \mathfrak{t}^*\left\{J + \frac{i}{2}\,\overline{J}\right\}\exp\left\{\iint \overleftarrow{\frac{\delta}{\delta\, J(u)}\, K_u}\; i\,\Delta^+\,(u - v)\; \overrightarrow{K_v\,\frac{\delta}{\delta\, J(v)}}\, du\, dv\right\}\mathfrak{t}\left\{J - \frac{i}{2}\,\overline{J}\right\} \tag{6.8}$$

Through formal calculation of the Δ^+-convolution via (4.4) the unrenormalised Feynman-Dyson perturbation solution (6.4) yields an unrenormalised perturbation expansion of $\mathfrak{r}$ (SYMANZIK, loc. cit.):

$$\mathfrak{r}\{\overline{J}, J\} = \exp\left\{2\int \sin\left[\frac{1}{2}\, D_x\,\frac{\delta}{\delta\,\overline{D}_x}\right] L_w\,[\overline{D}_x]\, dx\right\} e^{\overline{J}\,\Delta^R J + 1/4\,\overline{J}\,\Delta^1\overline{J}}, \tag{6.9}$$

$$\overline{D}_x \equiv \frac{\delta}{\delta \overline{J}(x)}, \qquad D_x \equiv \frac{\delta}{\delta J(x)}.$$

The argument of the first exponential is a convenient notation of the expression

$$i\, L_w\left[\overline{D} - \frac{i}{2}\, D\right] - i\, L_w\left[\overline{D} + \frac{i}{2}\, D\right] = 2\sin\left[\frac{1}{2}\, D\,\frac{\delta}{\delta\,\overline{D}}\right] L_w\,[\overline{D}] \tag{6.10}$$

derived from (3.2). From (6.9) follow unrenormalised r-functions via the relation

$$\mathfrak{r}_x\{J\} = \bar{D}_x\, \mathfrak{r}\{\bar{J}, J\}\,|_{\bar{J}=0}\,. \tag{6.11}$$

The desired linearisation of (6.6) is effected if one writes it in the form

[$\bar{O}$]
$$\theta\,(y-x)\,\bar{D}_x\,D_y\,\mathfrak{r}\{\bar{J}, J\}\,|_{\bar{J}=0} = 0\,, \tag{6.12}$$

where this equation is to be valid for all coefficients of the power series in the coupling constant. We prove (6.12) for the series (6.9) by consideration of the corresponding double graphs.

Double graphs have been first introduced by DYSON[17]) for the representation of the perturbation series of r-functions derived from Lagrangian theory. The name stems from the fact that in double graphs, not only the number of vertices and their connection structure is important, but in addition the subgraphs formed by the Δ^R-functions must be considered which contain an orientation structure induced by the time ordering of the arguments of the Δ^R.

A graphical notation for SYMANZIK's integrals (6.9) in the case of spin-0 quantum electrodynamics has been developed in the author's dissertation. It suffices for our present purpose to note that the Lagrangian functions of the first exponential correspond to the vertices of the double graphs; for every differentiation operator contained in them, a Δ^R- or Δ^1-line must be attached which leads either to another vertex, or to external currents $\bar{J}$, J.

DYSON's double graphs have the property of "T-structure", i.e. from every vertex issues exactly one Δ^R-line. For SYMANZIK's double graphs (6.9), we have the analogous property of $\mathfrak{R}$-*structure:* At every vertex begins at least one Δ^R-line. The proof follows immediately from the fact that every vertex in (6.9) contains at least one D-operator, which necessarily acts on the retarded end of a Δ^R-line.

We now prove that all contributions to the left side of (6.12) vanish. All possible contributions have exactly one external $\bar{J}$. If $\bar{J}$ is here end point of a Δ^1-line, then for graphs of finite perturbation order the $\mathfrak{R}$-structure implies at least one cycle of Δ^R-functions of the form

$$\Delta^R\,(x_1-x_2)\,\ldots\,\Delta^R\,(x_{n-1}-x_n)\,\Delta^R\,(x_n-x_1) \tag{6.13}$$

in the integrand. Because of the rule for expressions (6.5) these graphs give no contribution. $\bar{J}$ therefore is end point of a Δ^R-line, while all external J must be origins of Δ^R-lines. Such graphs which contain a J that is not connected to $\bar{J}$ through a chain of Δ^R-lines with consequent orientation, because of their $\mathfrak{R}$-structure must again contain a cycle (6.13) and give no contribution. The integrand of all possible contributions to the left side of (6.12) therefore contains a factor

$$\theta\,(y-x)\,\Delta^R\,(x-x_1)\,\ldots\,\Delta^R\,(x_{n-1}-x_n)\,\Delta^R\,(x_n-y) \tag{6.14}$$

and must again vanish because of the rule for expressions (6.5). We have thus completed the proof that FEYNMAN's integrals (6.4) formally satisfy the Bogoljubov equation (6.6) in all orders of perturbation theory.

7. τ-Equations and Renormalised Lagrangian Theory

Equivalent to the formal integral representation (6.4) for the unamputated τ-functions are the following formal integrals for the amputated τ-functions (see e.g. ref.[6])):

$$\left.\begin{aligned} & g^m \tau^{(m)}_{unren} (\bar{y}_1 \ldots \bar{y}_k) = i^{(m-k)} \frac{\delta}{\delta_{f(y_1)}} \cdots \frac{\delta}{\delta_{f(y_k)}} \int \cdots \int \theta(x_1 - x_2) \ldots \theta(x_{m-1} - x_m) \\ & \times \{L_w[f(x_1)] \underline{*}\, i \Delta^+ \underline{*} \{ \ldots \{L_w[f(x_{m-1})] \underline{*}\, i \Delta^+ \underline{*}\, L_w[f(x_m)]\} \ldots \}\} \, dx_1 \ldots dx_m \\ & \times t^{(0)}\{f\, i\, K_{x'}\} |_{f=0}, \end{aligned}\right\} \quad (7.1a)$$

$$F\{f\} \underline{*}\, i \Delta^+ \underline{*}\, G\{f\} \equiv F\{f\}\, e^{\overleftarrow{(\delta/\delta f)}\, i\, \Delta^+ \overrightarrow{(\delta/\delta f)}}\, G\{f\}. \quad (7.1b)$$

The question whether from the Feynman integrals (7.1) temperate distributions may be determined is called the renormalisation problem. We quote here known results (see e.g. BOGOLJUBOV and SHIRKOV[18])) in the form of

Lemma 5: In the integrand of (7.1), temperate distributions may be specified from the formal product expressions by addition of local Lorentz-invariant hermitian compensation terms*) to the Lagrangian L_w:

$$L_w[f(x)] \to L^{ren}_w[f(x)] = L_w[f(x)] + L_{comp}[f(x)], \quad (7.2a)$$

$$L_{comp}[f(x)] = \sum_{m=2}^{\infty} g^m \sum_n L^{(m)}_n [f(x)]. \quad (7.2b)$$

The coefficients C_{mni} of the monomials in L_{comp} are to compensate divergent integrals formally such that their difference δC_{mni} becomes finite; this manipulation may be defined e.g. as limit of a Pauli-Villars regularisation procedure[20]). The renormalised product expressions then contain a number of undetermined constants δC_{mni} which is finite in every order of perturbation theory**). –

After the renormalisation procedure, the existence of the integrals (7.1) is assured if condition (3'p) of the G.L.Z.-theorem is valid in every order of perturbation theory. This condition shall always be assumed in the following***).

Since the proof of (6.6) is valid for arbitrary L^{ren}_w, the perturbation theoretic Bogoljubov equation [condition (2'p)] is satisfied by the renormalised τ-functions of arbitrary Lagrange theories. Lemma 5 guarantees that these τ-functions are Lorentz-invariant temperate distributions; their symmetry follows from inspection of (7.1), so that condition (1p) is also completely satisfied.

Conversely, we now prove

*) If the renormalisation may be effected in such a way that in the formal power series (7.2b), the number of monomials in f contained in the coefficient of g^m is bounded uniformly in m, L_w is called renormalisable, otherwise non-renormalisable.

**) As additional renormalisation prescription, all vacuum graphs are neglected; this yields $t[0] = 1$ as assumed above.

***) BOGOLJUBOV and SHIRKOV (loc. cit.) have shown that this assumption is connected with the "adiabatic theorem"; the latter is treated e.g. in the encyclopedia article by G. KÄLLEN[19]).

Lemma 6: An arbitrary solution of the perturbation theoretic Bogoljubov-equation (5.6) can be represented by a renormalised Feynman-Dyson series

$$\mathfrak{t}\{I\} = \exp\left\{i \int L_{ren}\left[\frac{1}{i}\,\frac{\delta}{\delta\, I(x)}\right] dx\right\} \mathfrak{t}^{(0)}\{I\}\,. \tag{7.3}$$

Any non-trivial iteration solution of (5.6) needs non-vanishing $\sigma^{(m)}$ as starting-terms; for if all $\sigma^{(m)}$ are zero, then insertion of the $\tau^{(0)}$ derived from (6.3) yields $\tau^{(m)} = 0$ for $m > 0$. Since the $\sigma^{(k)}(x_1, \ldots, x_n)$ are real Lorentz-invariant temperate distributions with support in $x_1 = x_2 = \ldots x_n$, they are of the form

$$\sigma^{(k)}(x_1, \ldots, x_n) = \sum_{p(1 \ldots n)} Q_n^{(m)}\left[\ldots, p_1\left(\frac{\partial}{\partial x^{(1)}_{\mu_1}}\right) \prod_{\varrho=2}^{n} p_\varrho\left(\frac{\partial}{\partial x^{(\varrho)}_{\mu_\varrho}}\right) \delta^4\,(x^{(1)} - x^{(\varrho)}), \ldots\right] \tag{7.4}$$

where the p are polynomials in the space-time derivative operators, and the Q_n are real, Lorentz-invariant, and linear in arguments of the type indicated. Then if we define

$$\mathfrak{l}_n^{(k)}[f(x)] = Q_n^{(k)}\left[\ldots, \prod_{\varrho=1}^{n}\left[p_\varrho\left(\frac{\partial}{\partial x_{\mu_\varrho}}\right) f(x)\right], \ldots\right] \tag{7.5}$$

and if $\sigma^{(j)} = 0$ for $j = 0, 1, \ldots, k-1$, the lemma is proved for the lowest nontrivial order since

$$\mathfrak{t}^{(k)}\{I\} = i \int \sum_n \mathfrak{l}_n^{(k)}\left[\frac{\delta}{\delta I(x)}\right] dx\; \mathfrak{t}^{(0)}\{I\}\,. \tag{7.6}$$

The remainder of the proof follows by induction. It may now be assumed that for an arbitrary solution of (5.6), (7.3) is valid up to the order $m - 1$:

$$\mathfrak{t}^{(k)}\{I\} = \frac{1}{k!}\,\frac{\partial^k}{\partial g^k} \exp\left\{i \int L_{ren}^{(k)}\left[\frac{1}{i}\,\frac{\delta}{\delta\, I(x)}\right] dx\right\} \mathfrak{t}^{(0)}\{I\}\big|_{g=0}\,; \quad k = 0, 1, \ldots, m-1\,. \tag{7.7}$$

$L_{ren}^{(k)}$ is a renormalised Lagrangian of maximal degree k in the coupling constant g. Let $n_{min} + 2$ be defined as the smallest number of arguments for which the term in curly brackets in equation (5.6) does not vanish. To the Lagrangian $L_{ren}^{(m-1)}$ describing the solution in $(m-1)$th order, we now add a set of renormalisation terms of m-th order; $\tau^{(m)}(x\, y\, x_1 \ldots x_{n_{min}})$ calculated from (7.3) with the resulting $L_{ren}^{(m)}$ is then, because of the validity of (6.6) for arbitrary L_{ren}, equal to the expression calculated from (5.6) after insertion of the perturbation orders (7.7), up to an uncertainty of the form $\sigma^{(m)}(x, y, x_1, \ldots x_{n_{min}})$. After addition to $L_{ren}^{(m)}$ of the Lagrange term $\mathfrak{l}_{n_{min}+2}^{(m)}$ corresponding to $\sigma^{(m)}(x, y, x_1, \ldots x_{n_{min}})$, the induction step has been proved for $n = n_{min}$; it is proved in like manner for arbitrary n by iteration with the aid of all $\tau^{(m)}$ of a lesser number of arguments. Lemma 6 has thus been proved.

We collect the results of this section in the following

Theorem 1: The renormalised perturbation series of the τ-functions of an arbitrary local Lorentz-invariant Lagrangian formalism satisfy the perturbation theoretic Bogoljubov τ-equations as well as condition (1′p) in every order. Conversely, every solution of the perturbation theoretic Bogoljubov-equations satisfying conditions (1′p) may be represented by a renormalised local Lorentz-invariant Lagrangian formalism. Condition (3′p) has been assumed throughout. –

8. Consequences of the Equivalence Theorems

For arbitrary Lagrangian theories, renormalised perturbation series of the r-functions may be derived from the renormalised $\tau^{(m)}$-functions (assuming condition (3′p)) via the Δ^+-convolution (5.8), the result of which is a renormalised Dyson-Symanzik-prescription

$$\sum_{m=0}^{\infty} g^m \, \mathfrak{r}_x^{(m)}\{J\} = \bar{D}_x \exp\left\{2 \int \sin\left[\frac{1}{2} D_y \frac{\delta}{\delta \bar{D}_y}\right] L_{ren}[\bar{D}_y] \, dy\right\} e^{\bar{J} \Delta^R J + 1/4 \bar{J} \Delta^1 \bar{J}} \Big|_{\bar{J}=0}. \tag{8.1}$$

Theorem 1 and lemma 4 therefore imply:

Theorem 2: Perturbation expansions of the r-functions determined from a renormalised Dyson-Symanzik-prescription (8.1) of an arbitrary local Lorentz-invariant Lagrangian theory are a solution of the perturbation theoretic Peierls-equation (5.5) and satisfy condition (1p), where condition (3p) has been assumed. Conversely, every set of r-functions satisfying the r-equation (5.5) and conditions (1p), (3p) may be represented by a renormalised Dyson-Symanzik-prescription of a local Lorentz-invariant Lagrangian theory.

From this theorem follows immediately by lemma 3:

Theorem 3: If a perturbation series (5.1) of an interacting field $A(x)$ is formed by inserting the perturbation theoretic $r^{(m)}$-functions of an arbitrary local Lorentz-invariant Lagrangian formalism into the expansion (4.7):

$$A(x) = \sum_{m=0}^{\infty} g^m \sum_{n=1}^{\infty} \int \ldots \int r^{(m)}(x; x_1 \ldots x_n) : A^0(x_1) \ldots A^0(x_n) : dx_1 \ldots dx_n \tag{8.2}$$

where $A^0(x)$ is a free field with cyclic vacuum vector Ω, and condition (3p) is assumed to hold for the $r^{(m)}$, then $A(x)$ satisfies the perturbation theoretic L. S. Z.-axioms with $A_{in}(x) = A^0(x)$. – If on the other hand a perturbation expansion (8.2) of the field operator satisfies the perturbation theoretic L. S. Z.-axioms, then the $r^{(m)}$ may be represented by a Dyson-Symanzik-series (8.1) derived from a local Lorentz-invariant Lagrange theory*). –

It is noted that the partial sums (5.3) of the field operator will not in general satisfy the locality postulate (2.2). However, the perturbation theoretic locality (5.4) proved in theorem 3 suffices to prove the locality condition for perturbation theoretic Wightman functions; since further the spectral condition for $A(x)$ follows from the one for $A_{in}(x)$, we have as

*) J. Rzewuski[21]) derives a similar result for all Wightman field theories where certain existence assumptions for functionals hold. It seems that the justification of these assumptions by the results of K. Symanzik[5]) must be supplemented by the postulate of convergence of the perturbation series.

Corollary: If perturbation orders of the Wightman functions are constructed from the perturbation expansion (5.1) of a field operator calculated by an arbitrary local Lorentz-invariant Lagrangian formalism:

$$\mathfrak{W}^{(m)}(x_0 \ldots x_n) = \sum_{m_0 + \cdots + m_n = m} (\Omega, A^{(m_0)}(x_0) \ldots A^{(m_n)}(x_n)\, \Omega) \tag{8.3}$$

then each $\mathfrak{W}^{(m)}$ satisfies all linear conditions of Wightman's theorem[22]).

Acknowledgement

The work in this paper is taken from the author's dissertation at the E.T.H. Zürich. To my teacher, Prof. Dr. R. JOST, I should like to express my respectful gratitude for his steady encouragement and challenging criticism. I am very grateful to Dr. O. STEINMANN for several helpful discussions and valuable advice. Further thanks go to Prof. Dr. H. ARAKI and to Dr. D. RUELLE for instructive remarks.

Bibliography

1) H. LEHMANN, K. SYMANZIK, W. ZIMMERMANN, Nuovo Cimento *1*, 205 (1955); *ibidem 6*, 319 (1957).
2) L. SCHWARTZ, *Théorie des Distributions*, 2nd ed. Paris 1957.
3) O. STEINMANN, *Zur Definition der retardierten und zeitgeordneten Produkte* (Helv. Phys. Acta, to appear).
4) Cf. the review article by I. M. GELFAND and A. M. YAGLOM, Journ. Math. Phys. *1*, 48 (1960); and also the appendix of the dissertation of K. SYMANZIK, Göttingen 1954 (unpublished) and ref. 5.
5) K. SYMANZIK, Z. f. Naturforschung *9a*, 809 (1954).
6) P. CURTIUS, Dissertation E.T.H. Zürich 1962 (unpublished).
7) N. N. BOGOLJUBOV and D. V. SHIRKOV, *Introduction to the Theory of Quantised Fields*, New York 1959, chapter III.
8) R. E. PEIERLS, Proc. Roy. Soc. *A 214*, 143 (1952).
9) B. V. MEDVEDEV, *Axiomatic Method and Perturbation Theory*, Dubna preprint 1961.
10) K. NISHIJIMA, Progr. Theor. Phys. *17*, 765 (1957).
11) K. NISHIJIMA, Phys. Rev. *119*, 485 (1960).
12) V. GLASER, H. LEHMANN, W. ZIMMERMANN, Nuovo Cimento *6*, 1122 (1957).
13) H. M. FRIED, *Axiomatic Perturbation Theory for Retarded Functions*, Princeton University preprint 1962.
14) J. SCHWINGER, Proc. Nat. Acad. Sci. *37*, 452, 455 (1951).
15) The formal perturbation sums used here have been found independently 1952 by several authors. The literature is quoted in ref. 5.
16) K. SYMANZIK, Journ. Math. Phys. *1*, 249 (1960); K. SYMANZIK, in *Lectures in Theoretical Physics III* (W. BRITTIN, ed.), New York 1961.
17) F. J. DYSON, Phys. Rev. *82*, 428 (1951); *ibidem 83*, 608 (1951).
18) Ref. 7 chapter IV; references to older work, especially by E. C. G. STÜCKELBERG, are also found here.
19) G. KÄLLEN, Handbuch d. Physik, Vol. V, 1 (S. FLÜGGE, ed.).
20) W. PAULI and F. VILLARS, Rev. Mod. Phys. *21*, 434 (1949).
21) J. RZEWUSKI, CERN preprint 1962.
22) A. S. WIGHTMAN, Phys. Rev. *101*, 860 (1956).