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Space time reflexions, light quanta and heavy bosons

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Summary. A new approach to space time reflexions leads to the introduction of a new angular space in terms of which isobaric spin and related quantities find a simple explanation. On the basis of this new treatment a wave equation is presented which contains as specific cases the Maxwell equations, and the wave equations for π- and K-mesons. It also predicts the existence of a further particle with mechanical spin 1, with the same isobaric spin properties as K-mesons but with a larger rest mass.

1. Introduction

During a number of years reflexions of various kinds—space, time, charge—have been considered as very important for an understanding of fundamental particles. I feel that the usual treatment of such reflexions is open to serious criticism and I shall show that an appropriate change leads to the introduction of a new angular space in terms of which reflexions can be considered as special cases of continuous transformations. This new space offers an understanding of isobaric spin and related quantities and possibly also of rest mass. The great concern which W. Pauli showed for the treatment of reflexions makes it appropriate to discuss these new possibilities in the present memorial issue.

Consider first as a very simple example a two dimensional space with a coordinate frame x, y and an irregular triangle in it described by the coordinates of three points (x_1, y_1), (x_2, y_2), (x_3, y_3). A rotation replaces these three by three different coordinate pairs (x'_k, y'_k). Such a transformation can be interpreted in two different ways as is well known from geometry: (i) the triangle has not been moved, but the frame has been rotated; (ii) the frame remains the same but the triangle has been moved. The first interpretation has no physical (geometric) meaning in terms of the figure for a coordinate system is quite an arbitrary device. The second interpretation, however, has a very definite physical meaning connected with the displacement of the figure in space. To have a closer analogy with field equations we replace the triangles by the three straight lines forming it. They are described by three equations between y and x, $a_k x + b_k y + c_k = 0$. Rotation of the coordinate frame by an angle γ

(first interpretation) corresponds to a replacement of (x, y) by (x', y'), say, $x = x' \cos\gamma - y' \sin\gamma$; $y = x' \sin\gamma + y' \cos\gamma$. The appropriate motion of the triangle on the other hand corresponds to a replacement of a_k, b_k by $a'_k = a_k \cos\gamma + b_k \sin\gamma$; $b_k = - a_k \sin\gamma + b_k \cos\gamma$ (second interpretation). Carried out together the two transformations leave the form of the equations invariant.

Consider now a reflexion in which each x coordinate is replaced by its negative. According to the first interpretation we simply have replaced the original frame by another one. Since a frame is an arbitrary device such a change is, of course, always possible. The second, physical, interpretation is, however, no longer possible in a simple manner unless we extend the whole mode of description of the triangle. One such possibility would be the introduction of 'internal' coordinates permitting the triangle to be turned inside out (or rather its two dimensional analogue). Another possibility arises, however, if we permit the triangle to leave the $x - y$ plane by rotating it around the y-axis. This requires the introduction of a new dimension, an angle θ. The case $\theta = 0$ then would correspond to the original triangle, and $\theta = 180°$ to its mirror image. Invariance of the above three equations under reflexion thus involves (i) replacement of the coordinate frame (x, y) by $(- x, y)$ and (ii) rotation of the triangle around the y axis by 180°, leading to the replacement of a_k by $- a_k$.

Interpretation of the angle θ in terms of the $x - y$ plane suggests a formal connection with Pauli spin operators. The cases $\theta = 0$ and $\theta = 180°$ describing the original triangle and its reflected one would correspond to the two opposite directions of the spin. $\theta = 90°$ would then be interpreted as an appropriate mixture of these two cases.

Following the above discussion I feel that point transformations other than mere coordinate replacements should be considered as unphysical and should be replaced by continuous transformations through introduction of new angular coordinates (or of internal coordinates; this will not be done here). To introduce such a new description consider a four-vector field V_μ (space components V_k, time component V_0) in a Lorentz frame x_μ ($\mu = 1, 2, 3, 4$; $x_4 = i\, x_0$) and introduce three operators $\Pi_l(\Omega)$, $l = 1, 2, 3$ which depend on angles Ω in a way specified below. We then replace the V_μ by

$$(V_k, V_0) \to (\Pi_1\, V_k, i\, \Pi_2\, V_0) \tag{1.1}$$

and demand that

$$\Pi_1^2 = 1\,, \quad \Pi_2^2 = 1\,, \quad \Pi_3^2 = 1\,, \tag{1.2}$$

so that the length is given by

$$(\Pi_1\, V_k)^2 + (i\, \Pi_2\, V_0)^2 = V_k^2 - V_0^2 = V_k^2 + V_4^2\,. \tag{1.3}$$

Secondly we demand that the Π_l are invariant under continuous Lorentz transformations to that Π_1 is always connected with space, $i\,\Pi_2$ with time components. This is possible only if the relative velocity v entering the transformation is also replaced by an operator

$$v \rightarrow \Pi_3\, v \quad \text{i. e.} \quad (\Pi_3\, v)^2 = v^2\,. \tag{1.4}$$

A typical Lorentz transformation leading to relative motion in the 3-direction transforms V_μ into V'_μ. With the replacement (1.1), (1.4) and the requirement on Π_1 and Π_2 then

$$\left.\begin{aligned} (1-v^2)^{1/2}\,\Pi_1\, V'_3 &= \Pi_1\, V_3 - \Pi_3\, v(i\,\Pi_2)\, V_0 = \Pi_1\,(V_3 - v\, V_0) \\ (1-v^2)^{1/2}\, i\,\Pi_2\, V'_0 &= -\,\Pi_3\, v\, \Pi_1\, V_3 + i\Pi_2\, V_0 = i\Pi_2\,(-\,v\, V_3 + V_0) \end{aligned}\right\} \tag{1.5}$$

must hold. This is possible only if (cycl. means cyclic permutation of the suffixes 1, 2, 3)

$$i\,\Pi_3 = \Pi_1\,\Pi_2\,, \qquad \Pi_1\,\Pi_2 + \Pi_2\,\Pi_1 = 0\,, \quad \text{cycl.} \tag{1.6}$$

Thus, as a consequence of the consistency conditions expressed by equations (1.5) the Π_l must satisfy the Pauli anticommutation rules. One might, at first sight, expect that $\Pi_l = 1$ should satisfy consistency. This trivial possibility was excluded by the replacement of the time component V_0 by $i\,\Pi_2\, V_0$ which in view of $\Pi_2^2 = 1$ thus represents not an hermitean but an antihermitean operator. There is no reason why such operators should not be introduced provided that measurable quantities are described in terms of hermitean operators. It will be discussed in § 6 that this actually is the case. The introduction of the antihermitean operator $i\,\Pi_2$ has been postulated above with the purpose of obtaining a non trivial extension of the previous description of four-vector fields. It touches no doubt, however, on the very deep difference between space and time components whenever questions of reflexion are concerned.

Conditions (1.6) are, of course, fulfilled by Pauli matrices a_l (or $\boldsymbol{a}$) (dots and empty spaces represent zeros)

$$a_1 = \begin{pmatrix} . & 1 \\ 1 & . \end{pmatrix}, \qquad a_2 = \begin{pmatrix} . & -i \\ i & . \end{pmatrix}, \qquad a_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{1.7}$$

The conditions (1.6) are also satisfied by other matrices obtained from (1.7) by unitary transformations*). We therefore assume for the Π_l,

$$\Pi_l = s^{-1}\, a_l\, s \tag{1.8}$$

where s is unitary. In its most general form, s and hence Π_l depend on three real parameters which can be expressed in terms of Eulerian angles $\Omega = (\theta, \chi, \varphi)$ by

*) More general transformations will not be considered.

$$\left.\begin{aligned} s = e^{i\frac{\varphi}{2}a_3}\, e^{i\frac{\Theta}{2}a_2}\, e^{i\frac{\chi}{2}a_3} = e^{i\frac{\Theta}{2}a_2}\, e^{i\frac{\chi}{2}a_3}\, e^{i\frac{\varphi}{2}\pi_3} \\ 0 \leq (\chi, \varphi) \leq 2\pi\,, \quad 0 \leq \theta \leq \pi \end{aligned}\right\} \tag{1.9}$$

Eulerian angles are well known from the theory of spinning tops. They are defined in terms of an orthogonal frame (the external frame) in which the vector operator $\boldsymbol{a}$ has the three components a_l. The Eulerian angles then define three orthogonal unit vectors $\boldsymbol{u}_l$ (the inner frame; $\boldsymbol{u}_3$ is denoted as figure axis) such that

$$\Pi_l = (\boldsymbol{a}\, \boldsymbol{u}_l) \tag{1.10}$$

In the external frame the $\boldsymbol{u}_l$ have the following components,

$$\left.\begin{aligned} \boldsymbol{u}_1 &= (\cos\theta \cos\chi \cos\varphi - \sin\chi \sin\varphi\,, \quad \cos\theta \sin\chi \cos\varphi + \cos\chi \sin\varphi\,, \\ &\quad - \sin\theta \cos\varphi) \\ \boldsymbol{u}_2 &= (- \cos\theta \cos\chi \sin\varphi - \sin\chi \cos\varphi\,, \\ &\quad - \cos\theta \sin\chi \sin\varphi + \cos\chi \cos\varphi\,, \quad \sin\theta \sin\varphi) \\ \boldsymbol{u}_3 &= (\sin\theta \cos\chi\,, \quad \sin\theta \sin\chi\,, \quad \cos\theta)\,. \end{aligned}\right\} \tag{1.11}$$

Each Π_l can be considered as invariant under rotation of the external frame, being the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{u}_l$. Rotation then implies an unitary transformation acting on the $\boldsymbol{a}$ followed by a rotation of the $\boldsymbol{u}_l$ i.e. an appropriate change of the angles Ω. This invariance of the Π_l means that each replacement of Ω by Ω_1, say, can alternatively be expressed in terms of an appropriate unitary transformation. Thus unitary transformation by Π_2 which in view of $\Pi_2\, \Pi_1\, \Pi_2 = - \Pi_1$ leads to space reflexion is equivalent to the replacement of $\Omega = (\theta, \chi, \varphi)$ by $\Omega_1 = (\pi - \theta, \chi - \pi, \pi - \varphi)$. For with (1.11) one finds $\Pi_1(\Omega_1) = - \Pi_1(\Omega)$; $\Pi_2(\Omega_1) = \Pi_2(\Omega)$.

To illustrate the new treatment we investigate the vector field $\Pi_1\, p_k\, \psi$, $i\, \Pi_2\, p_0\, \psi$ of the momentum vector, where $\psi(x_\mu, \Omega)$ is a two component function and $p_\mu = - i\, \partial_\mu$. ψ can be classified in terms of Pauli spinors $\xi(\Pi)$, $\eta(\Pi)$ satisfying

$$\left.\begin{aligned} &\Pi_1\, \xi\,(\Pi) = \eta(\Pi)\,, \quad \Pi_1\, \eta(\Pi) = \xi(\Pi)\,; \quad \Pi_2\, \xi(\pi) = +\, i\, \eta(\Pi)\,, \\ &\Pi_2\, \eta(\Pi) = -\, i\, \xi(\Pi)\,; \quad \Pi_3\, \xi(\Pi) = \xi(\Pi)\,, \quad \Pi_3\, \eta(\Pi) = -\, \eta(\Pi)\,. \end{aligned}\right\} \tag{1.12}$$

From (1.8) and (1.7) therefore

$$\left.\begin{aligned} &\xi(\Pi) = s^{-1}\, \xi(a)\,, \quad \eta(\Pi) = s^{-1}\, \eta(a)\,, \\ &\qquad\qquad \text{where } \xi\,(a) = \binom{1}{0}\,, \quad \eta(a) = \binom{0}{1} \end{aligned}\right\} \tag{1.13}$$

Let now

$$\left.\begin{aligned} \psi_+(x, \Omega) &= \frac{1}{\sqrt{2}} (\xi(\Pi) + \eta(\Pi))\, e^{iK_\mu x_\mu}\,, \\ \psi_-(x, \Omega) &= \frac{1}{\sqrt{2}} (\xi(\Pi) - \eta(\Pi))\, e^{iK_\mu x_\mu}\,. \end{aligned}\right\} \quad (1.14)$$

The two cases ψ_+ and ψ_- correspond thus to vector fields with space components $+ K_k \psi_+$ and $- K_k \psi_-$ respectively. A similar separation of the time component into a forward $(+ K_0)$ and backward $(- K_0)$ field is not possible. In fact $\xi(\Pi) \pm i\,\eta(\Pi)$ which would diagonalise $i\,\Pi_2$ has imaginary eigenvalues which indicates that for a single vector field the decision whether the time component runs forward or backward is not a measurable quantity. The case of two vector fields is quite different, as will be shown in the following (in particular § 6). Here a measurable quantity exists which compares the direction of the time components of the two fields. It measures whether for the two these directions are equal, or opposite.

Finally, we show that space reflexion represents now a special case of the continuous unitary transformation by

$$e^{i\frac{\vartheta}{2}\Pi_2} = \cos\frac{\vartheta}{2} + i\,\Pi_2 \sin\frac{\vartheta}{2}\,. \quad (1.17)$$

It yields

$$e^{i\frac{\vartheta}{2}\Pi_2}\, \Pi_1\, e^{-i\frac{\vartheta}{2}\Pi_2} = \Pi_1 \cos\vartheta + \Pi_3 \sin\vartheta \quad (1.18)$$

and

$$e^{i\frac{\vartheta}{2}\Pi_2}\, \psi_\pm = \cos\frac{\vartheta}{2}\,\psi_\pm + \sin\frac{\vartheta}{2}\,\psi_\mp\,. \quad (1.19)$$

Clearly $\vartheta = 180°$, transforms Π_1 into $-\Pi_1$ and hence reflects the space component $\Pi_1\, p_k$.

In the following sections the ideas discussed here will be applied to the wave equation of bosons. On some simple assumptions it will be seen that from a single wave equation we obtain as special solutions the Maxwell equations, the equations for π- and K-mesons together with the correct isobaric spin assignments. We also predict existence of a particle with mechanical spin 1, with equal isobaric spin properties as K-mesons, but with larger rest mass.

Some of the following developments (§§ 2 and 3) have been presented before[1]) though in a less systematic form.

2. The Wave Equation of Bosons

Relativistic wave equations are based on the identity ($\varkappa$ = rest mass, units $\hbar = 1$, $c = 1$)

$$p_\mu\, v_\mu + \varkappa = p_k\, v_k - p_0\, v_0 + \varkappa = 0 \quad (2.1)$$

Here v_μ is the four-velocity which in contrast to classical physics is defined independently of the four-momentum p_μ in terms of certain operators, namely

$$p_\mu = -i\partial_\mu \ (2 \cdot 2)\,, \quad v_\mu = i\beta_\mu\,. \tag{2.3}$$

For bosons, following KEMMER[2]) the β_μ satisfy the algebraic relations

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\rho + \delta_{\rho\nu} \beta_\mu\,. \tag{2.4}$$

In contrast to KEMMER, however, the β_μ will be defined more explicitely in terms of two pairs of Pauli spins, ϱ_k, ϱ'_k; σ_k, σ'_k by

$$\beta_k = \frac{1}{2}(\varrho_1 \sigma_k + \varrho'_1 \sigma'_k)\,, \quad \beta_4 = \frac{1}{2}(\varrho_2 + \varrho'_2) \tag{2.5}$$

They are thus 16×16 matrices.

Our new treatment of four vectors requires the replacements (1.1) for both v_μ and p_μ. The independent definition of these quantities requires two sets of $\Pi_l(\Omega)$'s, say $\Pi_l(\Omega)$ and $\Pi'_l(\Omega')$ so that

$$p_k \to \Pi_1(\Omega)\, p_k\,, \quad p_0 \to i\,\Pi_2(\Omega)\, p_0 \tag{2.6}$$

$$v_k \to \Pi'_1(\Omega')\, v_k\,, \quad v_0 \to i\,\Pi'_2(\Omega')\, v_0\,. \tag{2.7}$$

Furthermore, since the length of a four vector is defined by (1.3) the product of two four vectors is given by

$$\left.\begin{aligned} p_\mu v_\mu &\to (\Pi_1 p_k)(\Pi'_1 v_k) + (i\,\Pi_2 p_0)(i\,\Pi'_2 v_0) = \\ &= \Pi_1 \Pi'_1 p_k v_k - \Pi_2 \Pi'_2 p_0 v_0 = \Pi_1 \Pi'_1 p_k v_k + \Pi_2 \Pi'_2 p_4 v_4\,. \end{aligned}\right\} \tag{2.8}$$

Making now the replacements (2.8), (2.2), (2.3), (2.5) in (2.1) leads to the wave equation

$$B_\mu \partial_\mu + M, \Psi = 0 \tag{2.9}$$

where

$$\left.\begin{aligned} B_k &= \frac{1}{2}\Pi_1(\Omega)\,\Pi'_1(\Omega')\,(\varrho_1 \sigma_k + \varrho'_1 \sigma'_k)\,, \\ B_4 &= \frac{1}{2}\Pi_2(\Omega)\,\Pi'_2(\Omega')\,(\varrho_2 + \varrho'_2) \end{aligned}\right\} \tag{2.10}$$

Here the rest mass $\varkappa$ has also been replaced by a mass operator M which will be discussed below. The B_μ are thus 64×64 matrices but such a matrix representation will never be required. The adjoint equation to (2.9) is given by

$$\overline{\Psi}, \quad B_\mu \partial_\mu - M = 0 \tag{2.11}$$

where (Ψ^+ is the hermitean conjugate of Ψ)

$$\bar{\Psi} = \Psi^+ R_2 \tag{2.12}$$

and

$$R_1 = - \varrho_1 \varrho_1', \quad R_2 = - \varrho_2 \varrho_2', \quad R_3 = - \varrho_3 \varrho_3 \tag{2.13}$$

i.e.

$$R_1 R_2 = R_3, \quad R_1^2 = 1, \quad \text{cycl.} \tag{2.14}$$

It will be noticed that the wave equation depends on three pairs of Pauli spins, (Π_l, Π_l'), (ϱ_k, ϱ_k'), (σ_k, σ_k'). The solutions will be discussed in terms of appropriate spin functions. For this purpose we introduce for each pair Pauli spinors $\xi, \eta; \xi', \eta'$ similar to (1.12). In case of ambiguity they will be denoted by $\xi(a)$, $\xi(\Pi)$ etc., if they refer to Pauli matrices $\boldsymbol{a}, \boldsymbol{\Pi}$, etc. Three symmetrical spin pair functions Σ_l and one antisymmetric Σ_4, can then be introduced by

$$\left.\begin{array}{c} \Sigma_1 = - \frac{i}{\sqrt{2}} (\xi\xi' - \eta\eta'), \quad \Sigma_2 = - \frac{1}{\sqrt{2}} (\xi\xi' + \eta\eta'), \\ \Sigma_3 = \frac{i}{\sqrt{2}} (\xi\eta' + \eta\xi') \end{array}\right\} \tag{2.15}$$

and

$$\Sigma_4 = \frac{i}{\sqrt{2}} (\xi\eta' - \eta\xi'). \tag{2.16}$$

They are orthogonal and normalised. In case of ambiguity they are denoted by $\Sigma_\sigma(a)$, $\Sigma_\sigma(\Pi)$, $\Sigma_\sigma(\varrho)$, $\Sigma_\sigma(\sigma)$, if they refer to the pairs $\boldsymbol{a}, \boldsymbol{\Pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ respectively. They have been chosen such as to diagonalise the three R_k (2.13) or the corresponding Π, σ, a quantities—as shown in table 1.

Table 1

Eigenvalues of R_k

	Σ_1	Σ_2	Σ_3	Σ_4
R_1	1	−1	−1	1
R_2	−1	1	−1	1
R_3	−1	−1	1	1

It follows from the above that a total of $4^3 = 64$ orthogonal normalised spin functions exist. In a matrix representation each would be represented by a column matrix containing 63 zeros and a one, the latter in a different place for each of the 64 functions.

We consider now first the 16 functions $\Sigma(\varrho)\ \Sigma(\sigma)$ referring to the ϱ and σ's, i.e. to the β_μ alone. Under exchange of dashed and undashed operators we find 10 symmetrical functions Γ_s and six antisymmetric ones Γ_a. Of the latter $\Sigma_2(\varrho)\,\Sigma_4(\sigma)$ is of no interest because

$$\beta_\mu\,\Sigma_2(\varrho)\,\Sigma_4(\sigma) = 0 \tag{2.17}$$

for all four β_μ. The remaining five antisymmetric functions are

$$\left.\begin{aligned} \Gamma_a(k) &= \Sigma_4(\varrho)\,\Sigma_k(\sigma)\,, & k &= 1, 2, 3 \\ \Gamma_a(4) &= \Sigma_3(\varrho)\,\Sigma_4(\sigma)\,, & \Gamma_a(5) &= \Sigma_1(\varrho)\,\Sigma_4(\sigma) \end{aligned}\right\} \tag{2.18}$$

The ten symmetric functions will be denoted as follows

$$\left.\begin{aligned} \Gamma_s(E_k) &= \Sigma_1(\varrho)\,\Sigma_k(\sigma)\,, & \Gamma_s(H_k) &= \Sigma_2(\varrho)\,\Sigma_k(\sigma) \\ \Gamma_s(\phi_k) &= -\,\Sigma_3(\varrho)\,\Sigma_k(\sigma)\,, & \Gamma_s(\phi_4) &= \Sigma_4(\varrho)\,\Sigma_4(\sigma) \end{aligned}\right\} \tag{2.19}$$

The action of the β_μ on these sixteen functions can be seen in a simple way from 5×5 and 10×10 matrix representations denoted as $\beta_\mu(5)$ and $\beta_\mu(10)$ respectively. In the former $\Gamma_a(\beta)$, $\beta = 1, 2 \dots 5$, are column matrices with zero's except with a one at the $\beta - th$ row. This implies that

$$\left.\begin{aligned} \beta_1(5) &= \left(\begin{array}{c|c} & 1 \\ & \vdots \\ \hline 1\ \dots & \cdot \end{array}\right), & \beta_2(5) &= \left(\begin{array}{c|c} & \vdots \\ & 1 \\ & \vdots \\ \hline .\,1\,.\,. & \cdot \end{array}\right), \\ \beta_3(5) &= \left(\begin{array}{c|c} & \vdots \\ & 1 \\ & \cdot \\ \hline .\,.\,1\,. & \cdot \end{array}\right), & \beta_4(5) &= \left(\begin{array}{c|c} & \vdots \\ & \vdots \\ & -i \\ \hline \dots\ i & \cdot \end{array}\right). \end{aligned}\right\} \tag{2.20}$$

For the symmetric case we may represent the $\Gamma_s(\beta)$, $\beta = 1, 2, \dots, 10$ by column matrices with zeros except a one at the $\beta - th$ row, where E_k stands for $\beta = 1, 2, 3$; H_k for $\beta = 4, 5, 6$; ϕ_μ for $\beta = 7, 8, 9, 10$. The matrices obtained in this manner are identical with Kemmer's 10×10 matrices (his equation (53)) provided all signs are reversed. They will not be reproduced here.

A complete classification of the spin functions is obtained from the Γ_a and Γ_s by multiplication with the four $\Sigma(\Pi)$. It will be seen that this would lead to π-mesons and light quanta only. To obtain K-mesons as well it is necessary to introduce further operators in (Ω, Ω') space (denoted as isospace). It is of course possible to introduce operators in this space corresponding to angular momenta different from Pauli type of spin. At present, however, it will be necessary to introduce a momentum I around the figure axis only—though in future it may well be required to introduce more general quantities. Let thus

$$I = -\,i\,\partial_\omega + \frac{1}{4}\,(\Pi_3 - \Pi_3') = -\,i\,U^{-1}\,\partial_\omega\,U \tag{2.21}$$

which commutes with all Π_l, Π'_l. Here

$$\omega = \varphi - \varphi' \quad \text{i. e.} \quad \partial_\omega = \frac{1}{2}(\partial_\varphi - \partial_{\varphi'}) \tag{2.22}$$

and

$$U = e^{i\frac{\varphi}{2}\Pi_3} e^{i\frac{\varphi'}{2}\Pi'_3} = e^{i\frac{1}{4}\omega(\Pi_3 - \Pi'_3)} e^{i\frac{1}{4}(\varphi+\varphi')(\Pi_3+\Pi'_3)} . \tag{2.23}$$

Like the Π_l, Π'_l the operator I is invariant under rotation of the external frame.

In addition to the four functions $\Sigma_\sigma(\Pi)$ we shall require four symmetric functions $\varkappa_\sigma$,

$$\varkappa_l = \cos\omega\, \Sigma_l(\Pi)\,, \quad \varkappa_4 = -\sin\omega\, \Sigma_4(\Pi)\,, \tag{2.24}$$

and four antisymmetric functions ν_σ,

$$\nu_l = -\sin\omega\, \Sigma_l(\Pi)\,, \quad \nu_4 = \cos\omega\, \Sigma_4(\Pi)\,. \tag{2.25}$$

The symmetry refers to interchange of dashed and undashed operators and coordinates so that Σ_l and $\cos\omega$ are symmetric, Σ_4 and $\sin\omega$ antisymmetric. It will be noted that

$$I^2\, \Sigma_\varrho(\Pi) = 0\,, \quad I^2 \varkappa_\varrho = \varkappa_\varrho\,, \quad I^2\, \nu_\varrho = \nu_\varrho\,. \tag{2.26}$$

Wave functions Ψ can now be written as a sum over spin functions each multiplied by a real space-time function. This reality condition will be found to be of considerable importance. Furthermore we postulate that only those spin functions which are antisymmetric under exchange of all pairs of dashed and undashed operators should be used. This leads to four types of wave functions Ψ_m, Ψ_π, Ψ_κ, Ψ_ν. Here Ψ_m and Ψ_π are based on $\Sigma_4(\Pi)$ and $\Sigma_l(\Pi)$ respectively forming a π-singlet and triplet. From the postulate of antisymmetry then, Ψ_m depends on $\Sigma_4(\Pi)\, \Gamma_s$, and Ψ_π on $\Sigma_k(\Pi)\, \Gamma_a$, so that we can write

$$\left.\begin{aligned} \Psi_m = \Sigma_4(\Pi) \Big\{ \sum_{k=1}^{3} \big(\Gamma_s(E_k)\, E_k(x_\mu) - \Gamma_s(H_k)\, H_k(x_\mu) + \\ + \Gamma_s(\phi_k)\, \phi_k(x_\mu)\big) - \Gamma_s(\phi_4)\, \phi_4(x_\mu)\Big\} \end{aligned}\right\} \tag{2.27}$$

and

$$\Psi_\pi = \sum_{l=1}^{3} \sum_{\beta=1}^{5} \Sigma_l(\Pi)\, \Gamma_a(\beta)\, \Phi_{l\beta}(x_\mu) \tag{2.28}$$

where the space-time functions $E_k(x_\mu)$ etc., $\Phi_{l\beta}(x_\mu)$ are assumed to be real.

Two further wave functions Ψ_κ and Ψ_ν can be formed from the $\varkappa_\sigma$ and ν_σ, the former with the help of Γ_a, and the latter with Γ_s,

$$\Psi_\kappa = \sum_{\sigma=1}^{4} \sum_{\beta=1}^{5} \varkappa_\sigma \Gamma_a(\beta) X_{\sigma\beta}(x_\mu) \tag{2.29}$$

and

$$\Psi_\nu = \sum_{\sigma=1}^{4} \sum_{\beta=1}^{10} \nu_\sigma \Gamma_s(\beta) V_{\sigma\beta}(x_\mu) \tag{2.30}$$

again with real functions $X_{\sigma\beta}(x_\mu)$ and $V_{\sigma\beta}(x_\mu)$. We note with (2.26)

$$I^2 \Psi_m = 0\,, \quad I^2 \Psi_\pi = 0\,, \quad I^2 \Psi_\kappa = \Psi_\kappa\,, \quad I^2 \Psi_\nu = \Psi_\nu\,. \tag{2.31}$$

Isobaric Spin and Charge Operators

The various spin operators permit formulation of conservation laws from the wave equations (2.9) and (2.11). For this purpose let J_c be an operator independent of x_μ which commutes with the B_μ and with M. Then

$$\partial_\mu J_{c\mu} = 0 \quad \text{where} \quad J_{c\mu} = i\,\overline{\Psi}\, J_c B_\mu \Psi\,. \tag{2.32}$$

We shall also demand that J_c be symmetrical in all dashed and undashed operators. A total of sixteen such J_c operators exist which can be derived from the $(\Pi_l + \Pi_l') R_l$, the $I(\Pi_l - \Pi_l') R_l$ or from their products. Since according to (2.31) eigenvalues $I^2 = 0$ and $I^2 = 1$ of I^2 only are required it is possible to derive these sixteen J_c from three basic operators A_l ($\boldsymbol{A}$)

$$A_l = \frac{1}{2} R_l (\mu \Pi_l + \mu' \Pi_l') = \frac{1}{2} R_l \alpha_l \quad \text{(no sum)} \tag{2.33}$$

where

$$\left.\begin{aligned} &\mu = \frac{1 - I^2 + 2I}{1 + I^2}\,, \quad \mu' = \frac{1 - I^2 - 2I}{1 + I^2} \\ &\text{i. e.}\ \ \mu^2 = \mu'^2 = 1\,; \quad \mu = \mu' = 1 \ \textit{if}\ I = 0\,; \quad \mu = I\,, \\ &\qquad \mu' = -I \ \textit{if}\ I^2 = 1\,. \end{aligned}\right\} \tag{2.34}$$

The R_l (2.13) are required here so that A_l commutes with B_μ. From A_l we derive three operators $Q_l(\boldsymbol{Q})$ by

$$i\,\boldsymbol{Q} = \boldsymbol{A} \times \boldsymbol{A}\,, \quad \text{i. e.}\ \ Q_l = \frac{1}{2} R_l (\Pi_l + \Pi_l') = R_l q_l\,. \tag{2.35}$$

Besides these six operators A_l, Q_l ten further ones are obtained for J_c namely six E_{kl}, three Λ_{kl}, and $J_c = 1$. Here

$$E_{kl} = A_k A_l + A_l A_k - \delta_{kl} = \frac{1}{2} \mu \mu' R_k R_l (\Pi_k \Pi_l' + \Pi_l \Pi_k')\,, \tag{2.36}$$

and

$$\left.\begin{aligned} \Lambda_{kl} = \frac{1}{2} (A_k Q_l - A_l Q_k + Q_l A_k - Q_k A_l) = \\ \frac{1}{4} R_k R_l (\mu - \mu') (\Pi_k \Pi_l' - \Pi_l \Pi_k') \end{aligned}\right\} \tag{2.37}$$

Introduction of these ten operators J_c $(1, E_{kl}, A_{kl})$ into expression (2.32) for $J_{c\mu}$ leads after integration over spin variables to antisymmetric expressions in the real space-time functions entering Ψ. These expressions vanish if the space-time functions commute. These ten J_c can, therefore, be used to formulate the reality conditions in an invariant manner. They are also expected to be of use for the formulation the commutation rules between these real space time functions (cf. § 5).

The remaining six J_c namely Q_l and A_l lead in a similar manner to symmetric expressions and can therefore be used for the formulation of the two main conservation laws, conservation of electric charge and of the third component of isobaric spin. We note in this connection that Q_3 for instance commutes with A_3, but with none other of the six Q_l, A_l. From Q_l and A_l we define two vector operators $\boldsymbol{T}$ and $\boldsymbol{S}$ by

$$T_l = \frac{1}{2}\,(Q_l + A_l) = \frac{R_l}{1+I^2}\left(\frac{\Pi_l + \Pi_l'}{2} + I\,\frac{\Pi_l - \Pi_l'}{2}\right) \qquad (2.38)$$

and (use $I^3 = I$ which holds for $I^2 = 0, 1$)

$$S_l = \frac{1}{2}\,(Q_l - A_l) = \frac{R_l\, I^2}{1+I^2}\left(\frac{\Pi_l + \Pi_l'}{2} - I\,\frac{\Pi_l - \Pi_l'}{2}\right). \qquad (2.39)$$

These operators as well as $\boldsymbol{Q}$ satisfy the angular momentum relations

$$\boldsymbol{Q}\ x\ \boldsymbol{Q} = i\,\boldsymbol{Q}\,,\quad \boldsymbol{T}\ x\ \boldsymbol{T} = i\,\boldsymbol{T}\,,\quad \boldsymbol{S}\ x\ \boldsymbol{S} = i\,\boldsymbol{S} \qquad (2.40)$$

(provided $I^2 = 0, 1$) though the generating operator $\boldsymbol{A}$ does not do so. We shall interpret Q_3 and $\boldsymbol{T}$ as the operators for electric charge and isobaric spin. $\boldsymbol{S}$ will be denoted as isospin shift because $Q_3 = T_3 + S_3$. S_3 was originally introduced by HEISENBERG[3]) together with isobaric spin; $2\,S_3$ is the so called strangeness. In connection with these interpretations it will be remembered that the Π_l, Π_l', and I are invariant under rotations of the external frame (in Ω, Ω' space) in a similar way in which the angular momenta of a spinning top, referred to its axes of symmetry are invariant under rotation. Since the first and second components, Π_1, Π_2, were connected with reflexions it seems natural to use the third one for the definition of charge. This, however, is no compelling reason. It will be shown in the discussion § 6, that a modification of the mass operator introduced below can be given such that the third components Q_3 and T_3 only are conserved (as well as T^2).

Matrix representations of the J_c are frequently useful. For this purpose we note that the operators α_l and q_l introduced in (2.33) and (2.35) are given by

$$\alpha_l = \alpha_l(4) = I\,\frac{\Pi_l - \Pi_l'}{2}\,,\qquad if\ I^2 = 1 \qquad (2.41)$$

and

$$\alpha_l = \alpha_l\,(3+1) = \frac{\Pi_l + \Pi_l'}{2} = q_l = q_l(4) = q_l\,(3+1)\,, \quad if\ I^2 = 0 \tag{2.42}$$

Here $\alpha_l(4)$ denotes a 4×4 matrix; $\alpha_l(3+1)$ indicates a 4×4 matrix which can be decomposed into a 3×3 and a 1×1 matrix. The actual form of these matrices is obtained by representing the functions $\Sigma_\sigma(\Pi)$ (referring to $I^2 = 0$) $\varkappa_\sigma$ and ν_σ (referring to $I^2 = 1$) by four sets of column matrices with zeros in every row except the $\sigma - t\,h$ which contains a 1. We then find

$$\alpha_1(4) = \left(\begin{array}{ccc|c} & & & -i \\ & & & \cdot \\ & & & \cdot \\ \hline i & \cdot & \cdot & \cdot \end{array}\right), \quad \alpha_2(4) = \left(\begin{array}{ccc|c} & & & \cdot \\ & & & -i \\ & & & \cdot \\ \hline \cdot & i & \cdot & \cdot \end{array}\right), \quad \alpha_3(4) = \left(\begin{array}{ccc|c} & & & \cdot \\ & & & \cdot \\ & & & -i \\ \hline \cdot & \cdot & i & \cdot \end{array}\right), \tag{2.43}$$

and

$$\left.\begin{array}{c} \alpha_1\,(3+1) = q_1(4) \\ = q_1\,(3+1) \end{array} = \left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \\ \cdot & \cdot & -i & \\ \cdot & i & \cdot & \\ \hline & & & \cdot \end{array}\right), \right.$$

$$\left.\begin{array}{c} q_2(4) \\ = q_2\,(3+1) \end{array} = \left(\begin{array}{ccc|c} \cdot & \cdot & i & \\ \cdot & \cdot & \cdot & \\ -i & \cdot & \cdot & \\ \hline & & & \cdot \end{array}\right), \quad \begin{array}{c} q_3(4) \\ = q_3\,(3+1) \end{array} = \left(\begin{array}{ccc|c} \cdot & -i & \cdot & \\ i & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \hline & & & \cdot \end{array}\right). \right\} \tag{2.44}$$

The decomposition of $\alpha_l(3+1) = q_l(3+1) = q_l(4)$ is indicated by rimming.

It seems of interest to mention here that $\alpha_l(4)$ and $\alpha_l(3) = q_l(3)$ are a four and a three dimensional representation of an α-algebra, based on three generating elements α_l satisfying the same rules (2.4) as do the four β_μ. This algebra has a further three dimensional representation by $\alpha_l(3^-) = -\,q_l(3)$, and a one dimensional one, $\alpha_l(1) = (.)$.

Mass Operator. From Kemmer's investigations it follows that the $\beta_\mu(5)$ connected with antisymmetric spin functions Γ_a occurring in Ψ_π and Ψ_κ lead to particles with mechanical spin 0, while the $\beta_\mu(10)$ connected with Γ_s lead to spin 1 particles. An operator M can be designed such that its eigenvalue differs for the four wave functions (2.27)–(2.30) while it commutes with Q_3, T_3, T^2, and I^2. This does not completely determine the forms of M. An ad hoc definition is, therefore, proposed which will be seen to yield a correct mass ratio for π- and K-mesons:

$$M = M(\Pi) + M(\varrho, \sigma) \tag{2.45}$$

where (C and c are numerical constants)

$$M(\Pi) = C\,(1 + I^2)^2\,(T^2 + I^2) \tag{2.46}$$

and
$$M(\varrho, \sigma) = c \left(\frac{\varrho_3 + \varrho_3}{2}\right)^2 \sum_{k=1}^{3} \left(\frac{\sigma_k + \sigma'_k}{2}\right)^2 . \tag{2.47}$$

We note that $M(\varrho, \sigma)\, \Gamma_a = 0$.

The form of $M(\Pi)$ is such that in a simple manner it separates eigenvalues corresponding to different values of T^2 and I^2. $M(\varrho, \sigma)$ will be required to obtain the Maxwell equations. Clearly a number of other possibilities for M do exist. A particular modification leading to conservation of the third components Q_3, T_3 only and not of Q_1, Q_2, T_1, T_2 would be obtained by multiplication of (2.45) by $\Pi_3\, \Pi'_3$, cf. § 6.

3. Solutions

We shall now introduce the four wave functions (2.27)–(2.30) into the wave equation (2.9) using M from (2.45)–(2.47). In each case a number of wave equations for the real space time functions will be found after multiplication with appropriate spin functions and integration over the spin variables. The procedure can be simplified, however, by using matrix representations appropriate for the spin functions involved in the particular case.

Maxwell field Ψ_m. The wave function Ψ_m, (2.27) is proportional to $\Sigma_4(\Pi)$. Hence with (2.31), (2.46), (2.35) and (2.38)

$$I\, \Psi_m = 0\,, \quad M(\Pi)\, \Psi_m = 0\,, \quad Q_l\, \Psi_m = 0\,, \quad T_l\, \Psi_m = 0 \tag{3.1}$$

and with (2.47)

$$\left.\begin{aligned} M(\varrho, \sigma)\, \Gamma_s(E_k) = 2\, c\, \Gamma_s(E_k)\,, \quad & M(\varrho, \sigma)\, \Gamma_s(H_k) = 2\, c\, \Gamma_s(H_k)\,, \\ M(\varrho\sigma)\, \Gamma_s\, (\phi_\mu) &= 0\,. \end{aligned}\right\} \tag{3.2}$$

Thus, Ψ_m has one Π-component describing an isobaric spin singlet, and ten (ϱ, σ) components. Action of the β_μ on the Γ_s may be represented in terms of $\beta_\mu(10)$ and the ten resulting space time equations are identical with the Maxwell equations in the form

$$\left.\begin{aligned} \operatorname{grad}\ \phi_4(x_\mu) + \partial_0\, \phi\, (x_\mu) + 2\, c\, E(x_\mu) &= 0\,, \\ \operatorname{curl}\ \phi(x_\mu) - 2\, c\, H(x_\mu) = 0\,, \quad \operatorname{curl}\ H\, (x_\mu) - \partial_0\, E\, (x_\mu) &= 0\,, \\ - \operatorname{div} E\, (x_\mu) &= 0\,. \end{aligned}\right\} \tag{3.3}$$

The mass constant c thus simply gives a measure for the vector potential $\phi_\mu(x_\mu)$.

π-Mesons, Ψ_π. For the wave function Ψ_π (2.28) depending on $\Sigma_l(\Pi)$ clearly

$$I\, \Psi_\pi = 0\,, \quad M(\pi)\, \Psi_\pi = 2\, C\, \Psi_\pi\,, \quad M(\varrho, \sigma)\, \Psi_\pi = 0\,. \tag{3.4}$$

It represents an isobaric spin triplet where using (2.35), (2.38) and (2.42)

$$Q_l = T_l = R_l\, q_l\,(3)\,. \tag{3.5}$$

Here $q_l(3)$ is the 3×3 rimmed sub matrix of $q_l(3+1)$ in (2.44), and this matrix acts on the suffix l of the real space time functions $\Phi_{l\beta}(x_\mu)$. Since Ψ_π depends on the Γ_a, the β_μ according to (2.20) can be represented by matrices $\beta_\mu(5)$ acting on the suffix β of $\Phi_{l\beta}$. Action of the $\Pi_1\,\Pi_1'$ and $\Pi_2\,\Pi_2'$ which enter the wave equation by (2.10) can be obtained from table 1. Thus if we introduce matrices $\varepsilon_{kk}(4) = \varepsilon_{kk}(3+1)$ by

$$\left.\begin{aligned} &\varepsilon_{11}(4) = \varepsilon_{11}\,(3+1) = \left(\begin{array}{ccc|c} 1 & & & \\ & -1 & & \\ & & -1 & \\ \hline & & & 1 \end{array}\right), \quad \varepsilon_{22}(4) = \varepsilon_{22}\,(3+1) = \left(\begin{array}{ccc|c} -1 & & & \\ & 1 & & \\ & & -1 & \\ \hline & & & 1 \end{array}\right), \\ &\qquad\qquad \varepsilon_{33}(4) = \varepsilon_{33}\,(3+1) = \left(\begin{array}{ccc|c} -1 & & & \\ & -1 & & \\ & & 1 & \\ \hline & & & 1 \end{array}\right) \end{aligned}\right\} \tag{3.6}$$

and if we understand by $\varepsilon_{kk}(3)$ the rimmed 3×3 sub-matrices, then the wave equation becomes after integration over spin variables

$$\varepsilon_{11}(3)\;\beta_k(5)\;\partial_k + \varepsilon_{22}(3)\;\beta_4(5)\;\partial_4 - 2\,C\,, \quad \Phi(x_\mu) = 0 \tag{3.7}$$

where $\Phi(x_\mu)$ represents the $\Phi_{l\beta}(x_\mu)$ with the ε_{kk} acting on the three suffixes l and the $\beta_\mu(5)$ on the five suffixes β. From Kemmer's investigations it is known that the $\beta_\mu(5)$ described particles with zero mechanical spin; equation (3.7) contains three such particles, with eigenvalues of $Q_3 = T_3$ obtained by diagonalising $q_3(3)$ i.e. $(1, 0, -1)$. Clearly (3.7) describes π-mesons provided the constant C is chosen appropriately.

K-Mesons, Ψ_κ. The wave function Ψ_κ, (2.29), depends on the four $\varkappa_\sigma$, so that with (2.31), (2.46) and (2.47)

$$I^2\,\Psi_\kappa = \Psi_\kappa\,, \quad M(\Pi)\,\Psi_\kappa = 7\,C\,\Psi_\kappa\,, \quad M(\varrho, \sigma)\,\Psi_\kappa = 0\,. \tag{3.8}$$

It representents two isobaric spin doublets with (use (2.35), (2.38), (2.39), (2.41) and (2.42)),

$$\left.\begin{aligned} Q_l\,\Psi_\kappa &= R_l\,q_l(4)\,\Psi_\kappa\,, \quad T_l\,\Psi_\kappa = \frac{1}{2}\,R_l\,(q_l(4) + \alpha_l(4))\,\Psi_k, \\ S_l\,\Psi_\kappa &= \frac{1}{2}\,R_l\,(q_l(4) - \alpha_l(4))\,\Psi_k. \end{aligned}\right\} \tag{3.9}$$

Since Ψ_κ depends on the Γ_a the β_μ can be represented by the $\beta_\mu(5)$ and the wave equation becomes

$$\varepsilon_{11}(4)\;\beta_\kappa(5)\;\partial_\kappa + \varepsilon_{22}(4)\;\beta_4(5)\;\partial_4 - 7\,C\,, \quad X = 0 \tag{3.10}$$

where X represents the functions $X_{\sigma',\beta}(x_\mu)$ with the $\varepsilon_{kk}(4)$, $\alpha_l(4)$, $q_l(4)$ acting on the four suffixes σ, and the $\beta_\mu(5)$ on the five suffixes β. This equation thus describes four particles with zero mechanical spin. They have electric charge (1, 0, − 1, 0) and 3rd isobaric spin components (1/2, − 1/2, − 1/2, 1/2) as is seen by diagonalising $q_3(4)$ and $\alpha_3(4)$ and using (3.9). Equation (3.10) thus described K-mesons with mass 7 C. Its ratio 7 : 2 to the π-meson mass is in close agreement with the experimental value 966 : 273 for charged K- and π-mesons.

ν-Mesons, Ψ_ν. This case based on the wave function Ψ_ν (2.30), again leads to two isobaric spin doublets with the same charge and isobaric spin properties as K-mesons. The antisymmetry of the four ν_σ, however, entails the use of Γ_s instead of Γ_a, and hence of a description of the β_μ by $\beta_\mu(10)$. This wave function therefore describes particles with mechanical spin 1, by

$$\varepsilon_{11}(4)\ \beta_\kappa(10)\ \partial_\kappa + \varepsilon_{22}(4)\ \beta_4(10)\ \partial_4 - (7\ C + 2\ c\nu)\ , \quad V = 0\ . \tag{3.11}$$

Here V represents the functions $V_{\sigma',\beta}(x_\mu)$ with the $\varepsilon_{kk}(4)$, $q_l(4)$ and $\alpha_l(4)$ acting on the four suffixes σ and the $\beta_\mu(10)$ acting on the ten suffixes β. ν is a diagonal 10×10 matrix acting on β with 1 in the first six, 0 in the following four diagonal positions (cf. 3.2). Formation of the second order (in ∂_μ) equations shows that this particle has a mass

$$M_\nu = \left(7\ C\ (7\ C + 2\ c)\right)^{\frac{1}{2}}\ . \tag{3.12}$$

To derive this expression in a formal way from (3.11) it must be noticed that the $\beta_\mu(10)$ do not commute with the matrix ν, but as can be seen from the matrix representation

$$\beta_\mu(10)\ \nu + \nu\ \beta_\mu(10) = \beta_\mu(10)\ . \tag{3.13}$$

Thus, it has been shown that the wave equation (2.9) together with the postulate of 'spin' antisymmetry leads to the Maxwell equations (3.3) and to the wave equations (3.7) and (3.10) for π- and K-mesons respectively, together with the correct isobaric spin assignments. It also leads to a ν-meson which has unit mechanical spin, equal isobaric properties with the K-mesons, but a different mass.

Energy Momentum Tensor. The B_μ entering wave equation (2.9) satisfy the same algebraic rules (2.4) as do the β_μ. A symmetric energy momentum tensor $\Theta_{\mu\nu}$ can, therefore, be derived in the same manner as done by KEMMER provided it is observed that M is now an operator. Thus generalising from Kemmer's equation (19) we have (the commutator of $M(\varrho, \sigma)$ and $B_\mu\ B_\nu$ applied to Ψ vanishes)

$$\Theta_{\mu\nu} = \overline{\Psi}\, M\ (B_\mu\ B_\nu + B_\nu\ B_\mu)\ \Psi - \delta_{\mu\nu}\ \overline{\Psi}\, M\ \Psi\ . \tag{3.14}$$

In particular, since $\Pi_1 \Pi_1' \Pi_2 \Pi_2' = - \Pi_3 \Pi_3'$,

$$\Theta_{4\kappa} = - \overline{\Psi} M \Pi_3 \Pi_3' (\beta_4 \beta_\kappa + \beta_\kappa \beta_4) \Psi \tag{3.15}$$

and using $2 B_4^2 - 1 = - R_2$, $R_2^2 = 1$ and the definition (2.12) of $\overline{\Psi}$,

$$\Theta_{44} = - \Psi^+ M \Psi . \tag{3.16}$$

For each of the fields discussed above this leads to the well known expressions, e.g.

$$\Theta_{44} = - 2 c \left(E^2 (x_\mu) + H^2 (x_\mu) \right) \tag{3.17}$$

for the Maxwell case,

$$\Theta_{44} = - 2 C \sum_{l=1}^{3} \sum_{\beta=1}^{5} \Phi_{l\beta}^2 (x_\mu) \tag{3.18}$$

for π-mesons, etc.

If we demand that the energy density $- \Theta_{44}$ be positive then both C and c must be positive. If furthermore $C = c$ then $M_\nu \simeq 1100$ electron masses would follow.

4. Reflexions

In the discussion of § 1 we have demanded that reflexions other than coordinate replacements should be expressed as continuous transformations. This led to the introduction of the $\Pi_l(\Omega), \Pi_l'(\Omega')$ in the new (Ω, Ω') angular space (isospace). As a consequence through the factors $\Pi_1 \Pi_1'$ and $\Pi_2 \Pi_2'$ the wave equation (2.9) has four components for each one in the previous treatments. This wave equation contains automatically the reflected ones expressed by appropriate changes in the angles Ω or Ω'. Thus, as can be seen from (1.11) and (1.10), replacement of $\Omega = (\theta, \chi, \varphi)$ by $\Omega_1 = (\pi - \theta, \chi - \pi, \pi - \varphi)$ (leaving Ω' unaltered) replaces $\Pi_1(\Omega) \Pi_1'(\Omega')$ by $\Pi_1(\Omega_1) \Pi_1'(\Omega') = - \Pi_1(\Omega) \Pi_1'(\Omega')$ leaving $\Pi_2 \Pi_2'$ unaltered. It is therefore equivalent to replacing x_k by $- x_k$. Thus the wave equation remains invariant if simultaneously we replace x_k by $- x_k$ and Ω by Ω_1 provided $\Psi(x_k, \Omega)$ is also replaced by $\Psi(- x_k, \Omega_1)$ and if similar replacements are made in all definitions of charge, isobaric spin and other operators. We remember that transformations of the angles Ω, Ω' are equivalent to unitary transformations. The above case for example, is equivalent to unitary transformation by Π_2 for $\Pi_2(\Pi_1 \Pi_1') \Pi_2 = - \Pi_1 \Pi_1'$. Similar transformations can be made in connection with reflexion of x_4 or of all four x_μ. Transformations of this kind will be denoted as insignificant transformations. They demonstrate that the choice of the original frame (x_k, x_0) is arbitrary. We might as well have chosen $(- x_k, x_0)$. Quite a number of other arbitrary choices have been made, of course. One, for instance, is the choice of $+i$ in (1.1) which might have been replaced by $-i$. The choice between these two is equivalent to the choice of a right handed or a left handed Π_l system. Another arbitrary choice concerns the choice $+i$ in the definition of $\mathbf{Q}$ in (2.35). Insignificant transformations can always be derived such as to put into evidence that a certain choice of a coordinate frame may be replaced by another.

Significant Transformations. In contrast, significant transformations make statements concerning different physical states. As an example we consider charge conjugation. We consider the replacement $\Omega \rightarrow \Omega_1$ discussed above, or its equivalent, unitary transformation by Π_2, and supplement it by the unitary transformation by Π_2', i.e. we are interested in simultaneous replacements $\Omega \rightarrow \Omega_1$, $\Omega' \rightarrow \Omega_1'$ or in the equivalent unitary transformation by

$$C_2 = \Pi_2 \, \Pi_2' = C_2^{-1}. \tag{4.1}$$

Since both Π_2 and Π_2' are equivalent to replacing in the wave equation x_k by $-x_k$, C_2 leaves the wave equation invariant, but it transforms Ψ into

$$\Psi_c\,(x_\kappa, \Omega, \Omega') = C_2\,\Psi\,(x_\kappa, \Omega, \Omega')\,. \tag{4.2}$$

If now we were to replace in all definitions the Π_l, Π_l' by the transformed ones, e.g.

$$Q_3 \rightarrow Q_{3c} = C_2\,R_2 \frac{\Pi_3 + \Pi_3'}{2}\,C_2^{-1} = -\,Q_3 \tag{4.3}$$

then complete invariance would be obtained, e.g.

$$J_\mu = \overline{\Psi}\,Q_3\,B_\mu\,\Psi = \overline{\Psi}_c\,Q_{3c}\,B_\mu\,\Psi_c \tag{4.4}$$

and the transformation would be an insignificant one. To obtain a significant transformation we shall not transform the various definitions (4.3), etc., but make use of the fact that together with a certain $\Psi(x_\mu)$, *another* wave function, $\Psi_c(x_\mu)$ also solves the same wave equation. This Ψ_c then leads to a different physical state, namely one in which all electric charges are reserved, because

$$\overline{\Psi}_c\,Q_3\,B_\mu\,\Psi_c = \overline{\Psi}\,(C_2\,Q_3\,C_2)\,B_\mu\,\Psi = -\,\overline{\Psi}\,Q_3\,B_\mu\,\Psi \tag{4.5}$$

The physical significance of C_2 is that to each wave function Ψ describing certain types of particles another one Ψ_c is coordinated in which all electric charges and currents are reversed.

Charge conjugation can be considered as a special case of a number of continuous transformations like charge rotation

$$C_2(\vartheta) = e^{i\vartheta}\,e^{i\vartheta Q_2} = e^{i\vartheta}\,(1 + Q_2^2\,(\cos\vartheta - 1) + i\,Q_2 \sin\vartheta) \tag{4.6}$$

where ϑ is a parameter; $C_2 = C_2(\pi)$. This transformation too leaves the wave equation invariant; it rotates the vector operator $\mathbf{Q}$ by an angle ϑ

around the Q_2 axis. In view of definition (2.35) of $\boldsymbol{Q}$ this transformation involves not only the Π_l, Π_l' but also R_2, except when $\vartheta = \pi$. Another continuous transformation which leaves the wave equation invariant and contains C_2 as a special case is by

$$D_2(\vartheta) = e^{i\vartheta\Pi_2\Pi_2'}\,; \quad D_2\left(\frac{\pi}{2}\right) = i\, C_2\,. \tag{4.7}$$

Internal Parity Reflexion. This reflexion can be defined as a significant transformation which leaves the (mechanical) spin operator $1/2(\sigma_k + \sigma_k')$ invariant but reverses the energy flux θ_{4k} without affecting the charge. Clearly unitary transformation by $R_2 = -\varrho_2\,\varrho_2'$ serves the purpose as may be seen from (3.15) and (2.5). The transformation of the wave function

$$R_2\,\Psi = -\varrho_2\,\varrho_2'\,\Psi \tag{4.8}$$

affects the spin functions $\Sigma_\sigma(\varrho)$ only in a manner shown in table 1. It is, therefore, equivalent to multiplying the various space time functions entering Ψ by $+1$ or -1. Table 2 shows this change of sign for the Maxwell and the π-meson functions obtained with the help of (2.27), (2.28), (2.18) and (2.19).

Table 2

	E_k	H_k	ϕ_k	ϕ_4	Φ_{lk}	Φ_{l4}	Φ_{l5}
R_2	$-$	$+$	$-$	$+$	$+$	$-$	$-$

The result is identical with the well known parity assignments.

The above definition of internal parity reflexion loses its significance for particles without mechanical spin; yet a significance is usually attached to the internal parity of say π-mesons though this exhibits itself only when their interaction with nucleons is considered. I feel that one should conclude from this that boson fields contain internal variables which so far have not yet been recognised. In terms of these variables internal parity reflexion should describe the turning inside-out of a bosons as a continuous transformation. As a formal step in this direction we might assume that the Pauli matrices ϱ_k, ϱ_k' and σ_k, σ_k' entering by (2.5) the wave equation (2.9) have a physical meaning such that any representation can be chosen for them which is derived from (1.7) by unitary transformations. As in the case of the Π_l, Π_l' this involves angular spaces. For the σ_k, σ_k' the requirement is to some extent already satisfied by the form of the wave equation, containing them as $(\boldsymbol{\sigma}\,\boldsymbol{\partial})$ and $(\boldsymbol{\sigma}'\,\boldsymbol{\partial})$ only. A simultaneous unitary transformation of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ then is equivalent to a rotation of the ordinary x_k-frame; the angular coordinates connected with the $\boldsymbol{\sigma}$ are thus the angles in ordinary space. For the ϱ_k, ϱ_k' however, the requirement is not obviously satisfied because Lorentz transformations lead to a specific kind of unimodular transformation which do not include our requirements. The simplest possible extension of the definition of the ϱ_k which would permit internal parity reflexion could be obtained by defining similar to (1.10),

$$\varrho_l = (\boldsymbol{a}(\varrho)\, \boldsymbol{w}_l)\,, \quad \varrho_l' = (\boldsymbol{a}'(\varrho)\, \boldsymbol{w}_l) \tag{4.9}$$

where $\boldsymbol{a}(\varrho)$, $\boldsymbol{a}'(\varrho)$ are Pauli matrices of the form (1.7), and $\boldsymbol{w}_l$ are three orthogonal unit vectors defined in terms of certain Eulerian angles. For our purpose we may choose in particular $\boldsymbol{w}_2$ such that

$$\varrho_2 = a_2(\varrho)\,, \quad \varrho_2' = a_2'(\varrho) \tag{4.10}$$

which implies that ϱ_1 and ϱ_3 do not contain a_2. We may then define

$$\left.\begin{aligned} R_2(\vartheta) = e^{i\vartheta \frac{\varrho_2 + \varrho_2'}{2}} &= 1 + \left(\frac{\varrho_2 + \varrho_2'}{2}\right)^2 (\cos\vartheta - 1) + \\ &+ i\,\frac{\varrho_2 + \varrho_2'}{2}\,\sin\vartheta \end{aligned}\right\} \tag{4.11}$$

so that

$$R_2 = R_2(\pi) = -\,\varrho_2\,\varrho_2' \tag{4.12}$$

and

$$R_2\left(\frac{\pi}{2}\right) = \frac{1 - \varrho_2\,\varrho_2'}{2} + i\,\frac{\varrho_2 + \varrho_2'}{2}\,. \tag{4.13}$$

Transformation by $R_2(\vartheta)$ does not leave the wave equation invariant; invariance is obtained, however, by a simultaneous rotation of the $\boldsymbol{w}_l$ around the $\boldsymbol{w}_2$ axis. Rotation by 180° then becomes equivalent to a replacement of ∂_k by $-\,\partial_k$ in a manner similar to the example discussed in § 1. The turning inside out process in the Maxwell case for instance leaves H_k and ϕ_4 unaltered, while E_k and ϕ_k are rotated such that for $\vartheta = 90°$, $E_k \to \phi_k$, $\phi_k \to -\,E_k$ and for $\vartheta = 180°$, $E_k \to -\,E_k$, $\phi_k \to -\,\phi_k$. It would be desirable to go closer into this process than has been possible by the present formal description.

Finally transformation by

$$\left.\begin{aligned} P(\vartheta) = e^{i\frac{\vartheta}{2}(\varrho_2\,\Pi_2 + \varrho_2'\,\Pi_2)} &= 1 + \left(\frac{\varrho_2\,\Pi_2 + \varrho_2'\,\Pi_2'}{2}\right)^2 (\cos\vartheta - 1) + \\ &+ i\left(\frac{\varrho_2\,\Pi_2 + \varrho_2'\,\Pi_2'}{2}\right) \sin\vartheta \end{aligned}\right\} \tag{4.14}$$

should be mentioned. It leaves the wave equation invariant, and for $\vartheta = 180°$ becomes

$$P(\pi) = -\,\varrho_2\,\varrho_2'\,\Pi_2\,\Pi_2' = R_2\,C_2 \tag{4.15}$$

i.e. combined charge and internal parity transformation.

5. Reality Conditions and Quantisation

The space time functions entering the wave functions in (2.27)–(2.30) were all assumed real. This statement is not invariant under unitary transformations and it was mentioned in § 2 that an invariant formulation is possible in terms of vanishing currents,

$$\overline{\Psi}(x)\ J_{kl}\ B_\mu\ \Psi(x) = 0 \tag{5.1}$$

where the J_{kl} represents the ten operators (cf. (2.36, 2.37))

$$J_{kl} = E_{kl}\,, \quad \Lambda_{kl}\,, \quad 1\,. \tag{5.2}$$

In (5.1) integration over all spin variables is assumed to be carried out so that these expressions depend on x_μ only. Conditions (5.1) remain valid if Ψ is multiplied by a phase factor $\exp(i\,\vartheta\,Q_s)$ or $\exp(i\,\vartheta\,A_s)$ because transformation of any of the J_{kl} by these terms (e.g. $\exp.(-\,i\,\vartheta\,Q_s)\ J_{kl}$ $\exp(i\,\vartheta\,Q_s)$) leads to a linear combinatiom of the ten J_{kl}.

The quantisation of the space time functions contained in Ψ is closely connected with the reality conditions. To demonstrate this we consider first a wave function of a neutral spin zero particle,

$$\beta_k(5)\,\partial_k + \beta_4(5)\,\partial_4 + \varkappa\,, \quad \psi(x) = 0\,. \tag{5.3}$$

Here $\psi(x)$ has five real components $\psi_\beta(x)$ which satisfy (use 2.20)

$$\left.\begin{aligned} \partial_k\,\psi_5 + \varkappa\,\psi_k = 0\,, \quad - i\,\partial_4\,\psi_5 + \varkappa\,\psi_4 = 0\,, \\ \partial_k\,\psi_k + i\,\partial_4\,\psi_4 + \varkappa\,\psi_5 = 0\,. \end{aligned}\right\} \tag{5.4}$$

Now if $\overline{\psi}(x) = \psi^+(x)\ (1 - 2\,\beta_4^2)$, then $\overline{\psi}(x)\ \beta_\mu(5)\ \psi(x)$ is conserved. From (5.3) and (5.4) we then find

$$\left.\begin{aligned} &\varkappa\left(\overline{\psi}(x')\ \beta_\mu(5)\ \psi(x) + \overline{\psi}(x)\ \beta_\mu(5)\ \psi(x')\right) = \\ &= (\partial_\mu - \partial_\mu')\ (\psi_5(x')\ \psi_5(x) - \psi_5(x)\ \psi_5(x'))\,. \end{aligned}\right\} \tag{5.5}$$

For classical fields this expression vanishes. When $\psi_5(x')$ and $\psi_5(x)$ do not commute, however, then it vanishes only if $x = x'$. Conversely the condition that the left hand side of (5.5) should vanish for $x = x'$ requires that all ψ_β be real apart from a common phase factor.

Now quantization of a real field $\psi_5(x)$ requires (cf. PAULI[4]) eqns. (32) and (22))

$$\psi_5(x')\ \psi_5(x) - \psi_5(x)\ \psi_5(x') = [\psi_5(x'), \psi_5(x)] = i\ D\ (x - x', \varkappa) \tag{5.6}$$

where $D(x - x', \varkappa)$ represents the universal D function for rest mass $\varkappa$. Hence the condition

$$\varkappa\left(\overline{\psi}(x')\ \beta_\mu(5)\ \psi(x) + \overline{\psi}(x)\ \beta_\mu(5)\ \psi(x')\right) = i\ (\partial_\mu' - \partial_\mu)\ D\ (x - x', \varkappa) \tag{5.7}$$

expresses the reality as well as the quantization condition.

Complete quantisation of the wave function Ψ involves not only the space time coordinates x_μ but also the new angular coordinates. At present quantization of the space-time functions only will be considered, and consequently the four fields (2.27)–(2.30) will be treated separately. Let $\Psi_f(x)$ be one of these four fields ($f = m, \pi, \varkappa, \nu$) and let M_f be the respective eigenvalue of the mass operator, defined as the value entering the second order wave equation

$$\Box\,\Psi_f - M_f^2\,\Psi_f = 0 \tag{5.8}$$

i.e. (for $f = m$, (5.8) holds for $E(x_\mu)$ and $H(x_\mu)$ only)

$$M_m = 0\,, \quad M_\pi = 2\,C\,, \quad M_k = 7\,C\,, \quad M_\nu = (7\,C\,(7\,C + 2\,c))^{\frac{1}{2}}. \tag{5.9}$$

We can then tentatively generalise (5.7) to

$$\left.\begin{aligned}\overline{\Psi}_f(x')\,\frac{1}{2}\,(M B_\mu + B_\mu M)\,J_{kl}\,\Psi_f(x) + \overline{\Psi}_f(x)\,\frac{1}{2}\,(M B_\mu +\\ + B_\mu M)\,J_{kl}\,\Psi_f(x') = i\,\delta_{kl}\,(\partial_\mu - \partial'_\mu)\,D\,(x - x', M_f)\,.\end{aligned}\right\} \tag{5.10}$$

Integration over all spin variables on the left hand side is assumed. It will be remembered that J_{kl} commutes with both M and B_μ. Also (cf. (2.46, 2.47)) $M(\Pi)$ commutes with B_μ, but $M(\varrho, \sigma)$ does not do so in the case of particles with mechanical spin 1, as indicated in (3.13). The case $k = l$ is assumed to comprise for J_{kk} not only the three operators E_{kk}, but also the unity operator 1. In a matrix representation, the three E_{kk} are given as diagonal matrices (3.6). Together with 1 they can be combined into four matrices containing zeros except for a single diagonal element each. For spin 0 particles ($f = \pi$ or K) the four ($f = K$) or three ($f = \pi$) conditions arising from (5.10) with $k = l$ are then identical with (5.7) for each of the four (three) Π-components, apart from signs. The remaining conditions (5.10), $k \neq l$, show that different Π-components commute.

The case of spin 1 particles will be illustrated on the Maxwell field Ψ_m. This field has a single Π-component $\Sigma_4(\Pi)$. All J_{kl} conditions then either give $0 = 0$ (if $k \neq l$) or they become identical with $J_{kk} = 1$. Using (3.13) thus

$$\left.\begin{aligned}- c\,(\overline{\Psi}_m(x')\,\beta_\mu(10)\,\Psi_m(x) + \overline{\Psi}_m(x)\,\beta_\mu(10)\,\Psi_m(x')) =\\ = i\,(\partial_\mu - \partial'_\mu)\,D\,(x - x', 0)\,.\end{aligned}\right\} \tag{5.11}$$

Inserting for Ψ_m from (2.27) yields for $\mu = 1, 2, 3$,

$$\left.\begin{aligned}- c([E_1(x'), \phi_4(x)] + [E_1(x), \phi_4(x')] - [H_2(x'), \phi_3(x)] -\\ - [H_2(x), \phi_3(x')] + [H_3(x'), \phi_2(x)] + [H_3(x), \phi_2(x')]) =\\ = i\,(\partial_1 - \partial'_1)\,D\,(x - x', 0)\,; \quad \text{cycl}.\end{aligned}\right\} \tag{5.12}$$

and for $\mu = 4$

$$\left.\begin{aligned} - i\, c \sum_{k=1}^{3} \left([E_k(x'), \phi_k(x)] + [E_k(x), \phi_k(x')] \right) = \\ = i\, (\partial_4 - \partial_4')\, D\, (x - x', 0)\, . \end{aligned}\right\} \quad (5.13)$$

Expressing E_k and H_k by ϕ_μ through the field equations (3.3) leads to four conditions containing the ϕ_μ and their derivatives only. They can be fulfilled if

$$[\phi_\mu\, (x'), \phi_\nu\, (x)] = - \frac{2}{3}\, i\, D\, (x - x', 0)\, \delta_{\mu\nu} \quad (5.14)$$

with

$$\delta_{\mu\nu} = 0 \ \text{ if } \ \mu \neq \nu\, ; \quad \delta_{kk} = 1 \ \text{ if } \ \varkappa = 1, 2, 3\, ; \quad \delta_{44} = -1\, . \quad (5.15)$$

Further investigation will have to show whether (5.14) is the only conclusion that can be drawn from (5.13) and whether (5.10) requires modification.

6. Discussion

The developments described in this paper are based on a criticism of the usual methods of dealing with reflexions. Arising from this it was postulated that all reflexions except coordinate replacements should be treated as continuous transformations. This required the replacements of four-vectors by operators according to (1.1) and led to the introduction of a new angular space (isospace). The application of these ideas to bosons is based on the identity (2.1) and is dominated by a very important difference between classical and quantum mechanics. In classical relativistic theory the momentum four-vector p_μ is connected to the velocity four-vector v_μ through the rest mass $\varkappa$ by $p_\mu = \varkappa\, v_\mu$; in quantum mechanics the two are defined independently (cf. (2.2), (2.3)). The quantum mechanical definition of v_μ is, of course, closely correlated with Schrödingers Zitterbewegung which exists not only in the Dirac equation but also for bosons (it is longitudinal for spin zero bosons). The independence of definition of the p_μ and v_μ four-vectors has a profound influence on the wave equation when the replacements (2.6), (2.7) are made, replacing $p_k\, v_k$ by $\Pi_1\, \Pi_1'\, p_k\, v_k$, and $p_0\, v_0$ by $- (i\, \Pi_2)\, (i\, \Pi_2')\, p_0\, v_0$. For clearly the Π_l, Π_l' occur only as products $\Pi_l\, \Pi_l'$. Now it was demonstrated in § 1 that $\Pi_1\, p_k$ (or $\Pi_1'\, v_k$) describes in terms of a space frame x_k both, a vector field p_k (v_k) and its reflected $- p_k$ ($- v_k$). The operator $\Pi_1\, \Pi_1'\, p_k\, v_k$, however, measures by $\Pi_1\, \Pi_1'$ only whether both vector fields p_k and v_k refer to the same sign $+ p_k$ and $+ v_k$ or $- p_k$ and $- v_k$ (we say they have equal reflexion status) or whether they refer to opposite signs, $+ p_k$ and $- v_k$, or $- p_k$ and $+ v_k$. A distinction between the individual vector fields $+ p_k$ and $- p_k$, or between $+ v_k$ and $- v_k$ does thus not enter our boson wave equation. The quantity that enters is the distinction as to whether the two have equal or unequal reflexion status. A corresponding result holds for the time components p_0, v_0. It will be noticed in particular that although $(i\,\Pi_2)$ and $(i\,\Pi_2')$ are anti-hermitean, their product is hermitean and thus describes a measurable quantity. While a distinction between the two temporal directions in a single four-vector field is not a measurable quantity, in our theory, the distinction as to whether two four-vector fields have equal or opposite temporal direction (temporal reflexion status) is measurable in terms of the operator $-\Pi_2\,\Pi_2'$. It is of interest to note in this connection that the four function $\Sigma_\sigma(\Pi)$, (2.15), (2.16) which simultaneously diagonalise both $\Pi_1\,\Pi_1'$ and $\Pi_2\,\Pi_2'$ can be expressed in terms of the functions

$$\zeta_+ = \frac{1}{\sqrt{2}}\left(\xi(\Pi) + \eta(\Pi)\right), \quad \zeta_- = \frac{1}{\sqrt{2}}\left(\xi(\Pi) - \eta(\Pi)\right), \tag{6.1}$$

which have been used in (1.14). For

$$\left.\begin{aligned} \Sigma_1 &= -\frac{i}{\sqrt{2}}\left(\zeta_+\,\zeta_-' + \zeta_-\,\zeta_+'\right), & \Sigma_2 &= -\frac{1}{\sqrt{2}}\left(\zeta_+\,\zeta_+' + \zeta_-\,\zeta_-'\right), \\ \Sigma_3 &= \frac{i}{\sqrt{2}}\left(\zeta_+\,\zeta_+' - \zeta_-\,\zeta_-'\right), & \Sigma_4 &= \frac{i}{\sqrt{2}}\left(\zeta_+\,\zeta_-' - \zeta_-\,\zeta_+'\right). \end{aligned}\right\} \tag{6.2}$$

As a consequence of the introduction of the operators Π_l and Π_l' the wave equation corresponds now not only to the one classical identity $p_k\,v_k - p_0\,v_0 + \varkappa = 0$ but to four such identities corresponding to replacements of $p_k\,v_k$ and $p_0\,v_0$ by $\pm\,p_k\,v_k$ and $\pm\,p_0\,v_0$ with all four combinations of the $\pm$ signs. This is easily seen by using a diagonal matrix representation for $\Pi_1\,\Pi_1'$ and $\Pi_2\,\Pi_2'$, based on table 1. Classically these combinations of signs would be impossible because of the classical identity $p_\mu = \varkappa\,v_\mu$ according to which replacement of p_k by $-\,p_k$ entails a replacement of v_k by $-\,v_k$, leaving $p_k\,v_k$ unaltered.

Discussion of the conservation laws derived from wave equation (2.9) has led to the definition of operators for isobaric spin T_l (2.38) and for electric charge $Q_3 = R_3(\Pi_3 + \Pi_3')/2$, (2.35). It is of considerable interest to discuss the meaning of these operators in terms of the reflexion status of (p_k, v_k), i.e. in terms of eigenstates of $\Pi_1\,\Pi_1'$ and $\Pi_2\,\Pi_2'$ as described above. We shall in particular deal with the electric charge operator Q_3 and notice that in the matrix representation which diagonalises $\Pi_1\,\Pi_1'$ and $\Pi_2\,\Pi_2'$, the operator $(\Pi_3 + \Pi_3')/2$ is represented by q_3, (2.44). It acts only on the first two Π-components of the wave function corresponding to the fact that there are only two non zero types of electric charge (positive and negative). Our attention can thus be restricted to these two;

in a formal way they are obtained by operating with Q_3^2 on the wave equation. We then find the two equations,

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \beta_k \partial_k + \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \beta_4 \partial_4 + \varkappa\,, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{6.3}$$

because the matrix representations of $\Pi_1 \Pi_1'$ and $\Pi_2 \Pi_2'$ are given by $-\varepsilon_{11}$ and $-\varepsilon_{22}$ (eqn. 3.6) respectively. Here ψ_1 and ψ_2 stand for $\Phi_{1\beta}$, $\Phi_{2\beta}$ in the case of π-meson, or for $X_{1\beta}$, $X_{2\beta}$ in the case of K-meson; $\varkappa$ represents the respective mass. The $\pm \beta_k \partial_k$ and $\pm \beta_4 \partial_4$ show that the charged components correspond for p_μ, v_μ either (ψ_1) to equal temporal and opposite spacial reflexion status (i.e. both p_0, v_0 forward or both backward; $+ p_k$ combined with $- v_k$ or $- p_k$ with $+ v_k$) or (ψ_2) to opposite temporal and equal spacial reflexion status. The charge operator in terms of the two components ψ_1, ψ_2 is represented by

$$Q_3 \rightarrow R_3 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{6.4}$$

as obtained from (2.35) and (2.44). It can be brought to principal axes by unitary transformation by

$$G = \frac{1}{\sqrt{2}} \begin{pmatrix} R_2 & iR_1 \\ R_2 & -iR_1 \end{pmatrix}, \quad G\, Q_3\, G^{-1} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \tag{6.5}$$

which combines ψ_1 and ψ_2 into (ψ_1 and ψ_2 are real)

$$\left.\begin{aligned} G \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix}, \quad \psi &= R_2 \psi_1 + i\, R_1 \psi_2 = R_2 (\psi_1 + i\, R_3 \psi_2) \\ \psi^* &= R_2 \psi_1 - i\, R_1 \psi_2 \end{aligned}\right\} \tag{6.6}$$

where ψ, ψ^* satisfy the conventional Kemmer equation $\beta_\mu \partial_\mu + \varkappa, \psi = 0$ and its conjugate complex.

In our theory, therefore, a state with definite electric charge is described by a superposition of a wave function ψ_1 having equal temporal and opposite spacial reflexion status of p_μ, v_μ and a wave function $R_3 \psi_2$ having the opposite temporal and equal spacial reflexion status (combined with an internal space-time reflexion by R_3). In fact, electric charge is defined as the operator Q_3 causing an oscillation between these two reflexion states ψ_1, ψ_2. It is, therefore closely connected with space time reflexions.

At this point it seems worth pointing out that the four-current based on β_μ vanishes identically in view of the reality conditions, while currents based on $v_\mu = (i\, \Pi_1' \beta_k, - \Pi_2 \beta_4)$ are not conserved. Conserved non vanishing currents depend, of course, on v_μ but involve other operators like Q_3, T_3 as well. The electric current density (space component) for instance is from (2.32) and (2.10) obtained as $\overline{\Psi}(x)\, Q_3\, \Pi_1 \Pi_1'\, i\, \beta_k\, \Psi(x) =$

$= \overline{\Psi}(x)\, Q_3\, \Pi_1\, v_k\, \Psi(x)$. We notice (i) that this current contains $v_k = i\, \Pi_1'\, \beta_k$ only in the form $\Pi_1\, v_k = i\, \Pi_1\, \Pi_1'\, \beta_k$ and thus depends on $\Pi_1\, \Pi_1'$ and not on Π_1' alone. Secondly, it refers to a specific quantity Q_3 or T_3 which it carries. Similar remarks hold for the relation of the momentum $P_\mu = i \int \theta_{4\mu}\, d^3x$ to the operator p_μ.

Definition of the electric charge operator by Q_3 leads to the question of why it is that we choose the third component of $\boldsymbol{Q}$ for it. We note, of course, the invariance of the Π_l, Π_l' under rotations of the external frame in isospace, but no compelling reason has been given to use the third and not another component. A modification of the wave equation (2.9) can be given which does in fact lead to the desired result, namely replacement of (2.9) by

$$B_\mu\, \partial_\mu + \Pi_3\, \Pi_3'\, M\,, \quad \Psi = 0\,. \tag{6.7}$$

Clearly only Q_3 and T_3, but not Q_1, Q_2, T_1, T_2, lead now to conserved currents. It will be noticed that the energy momentum tensor $\theta_{\mu\nu}$ defined in (3.14) does not require modification. For $\partial_\nu\, \theta_{\mu\nu} = 0$ holds even when (2.9) is replaced by (6.7); on the other hand $-\theta_{44}$ remains positive, provided the constants C and c entering the operator M are positive. The reality conditions also remain unaltered.

The solutions of the wave equation involved three pairs of Pauli spin operators namely Π_l, Π_l'; ϱ_l, ϱ' and σ_l, σ_l'. Furthermore, in the case of the Π_l, Π_l', an orbital operator I was introduced. It was then postulated that Ψ should be antisymmetric under exchange of all pair variables. This led to two types of solutions depending on whether the eigenvalue of I^2 was 0 or 1. In the former case an isobaric spin triplet and a singlet results. The triplet relates to mechanical spin 0 particles (π-mesons) and the singlet to spin particles with zero mass, i.e. it results in the Maxwell equation. The case $I^2 = 1$ leads to four isobaric spin states both with $T_3 = \pm 1/2$ as required by K-mesons. This case yields besides K-mesons (mechanical spin 0) another two isobaric doublets with mechanical spin 1 and a larger mass than that of K-mesons (ν-particles).

The operator I is closely connected with the orbital angular momentum in isospace around the 3-axis (the figure axis). Like Π_3 and Π_3', I is invariant under rotation in isospace. The definition of an isobaric spin vector $\boldsymbol{T}$, such that $\boldsymbol{T} \times \boldsymbol{T} = i\, \boldsymbol{T}$ is satisfied was possible only for the eigenvalues 0 and 1 of I^2. The physical meaning of this restriction is not clear to me at the moment.

The mass operator M was introduced such that solutions with different isobaric spin have different rest mass. A very simple form of M led to nearly the correct mass ratio of π- and K-mesons. This form of M has not been derived, however, in a compelling way. Nevertheless, already at this stage introduction of the M-operator demonstrates an attitude towards an interpretation of rest mass which is very different from one which regards this quantity as resulting from field self energies. If this attitude is correct then interactions should be formulated in terms of an (improved) mass operator.

Note added in proof

1. By replacing in wave equation (2.9) the Kemmer operators β_μ by Dirac operators, the wave equation for π-mesons based on the symmetric $\Sigma_e(II)$ functions leads to the wave equation for the electron-neutrino field provided $M = m\, q_3^2$ is used. The wave function has then $3 \times 4 = 12$ components which with an appropriate reality condition provides 6 complex components as required by the 2-component neutrino, 4-component electron theory. Expression (2.35) can still be used for the electric charge operator Q_3, though R_3 is now a different operator, such that isospin can no longer be defined because (2.40) can not be fulfilled. This operator Q_3 and an appropriate operator for neutrino charge provide automatically the expressions required by hole theory. Electric charge conjugation is still given by (4.11) but parity reflexion can not be separated from neutrino charge conjugation.

2. Careful investigation reveals two types of Lorentz transformations: one which retains complete symmetry in isospace, leaving all Q_k (and T_k) invariant; and a second which leaves only Q_3 invariant and transforms charged and uncharged fields independent of each other.

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