1. Introduction

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THE FIXED POINT SET OF A FINITE GROUP ACTION ON A HOMOLOGY FOUR SPHERE

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1. Introduction

It is a classical result of P. A. Smith that a finite p-group acting on a finite dimensional complex with the Z/p homology of a sphere S^k has fixed point set Z/p homologically equivalent to some S^k with $k < n$.

This theorem cannot, in general, be extended to groups of more general type, even if one assumes much more restrictive hypotheses such as a smooth action on a manifold homeomorphic or diffeomorphic to a sphere.

In particular, for every odd $n \geqslant 5$, it is possible to use Brieskorn varieties to produce finite group actions with fixed point set not a homology sphere. Even if the fixed point set is a sphere it can be embedded in a non standard way; for an elementary discussion of this phenomenon see [18] and [15]. For other "strange" actions of groups on higher dimensional spheres and disks the reader is referred to [16].

In low dimensions it is harder to construct such examples, and it may be conjectured that finite group actions on spheres are equivalent or somewhat "close" to linear ones.

On S^2 the situation is the best possible, indeed according to [6], [13] and [8], every finite group of homeomorphisms of S^2 is topologically conjugate to a linear action.

On S^3 it is necessary to assume local linearity, otherwise pathologies such as horned spheres may arise, see [2].

A deep and difficult theorem, conjectured by Smith and proved by combining results of Thurston, Meeks and Yau and Bass, states that every smooth cyclic action on S^3 is conjugate to the linear one; a detailed account can be found in [15].

The first example of a smooth cyclic action on S^4 with fixed point set a knotted S^2 is in [12], for more information see [9] and [17], these actions are obviously not linear.

The aim of this paper is to prove that any locally linear orientation preserving action of a finite group on an homology four sphere has fixed point set homeomorphic to a sphere. In particular there are no one fixed point actions. Besides, if the fixed point set is S^0, it is proved that the local representations are conjugate.[1]) For a large class of actions, the proof is an elementary application of Smith's theory, using the fact that in dimensions $\leqslant 2$ homology spheres are topological spheres. In one remaining case, an action of the icosahedral group, a slightly more complicated argument is needed. This type of argument cannot be extended to dimension 3, as the example in [11] proves.

The motivation for this work came from the paper of Peter Braam and Gordana Matic [3] on group actions and instantons spaces. They prove that a smooth orientation preserving action of a group on a homology sphere whose fundamental group has no nontrivial representations in $SU(2)$ admits an even number of isolated fixed points and that they come in pair such that the representations around them are conjugate. Also, Furuta proved that there are no actions with one fixed point.

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2. Statement of the result

In the following "R-homology S^n" will mean a compact topological manifold whose homology with coefficients in the ring R is the same as that of S^n. (Of course in dimensions 0, 1, 2 such a manifold is homeomorphic to a sphere.) To unify some notation, the empty set will be considered a sphere of dimension -1, all actions will be assumed effective.

THEOREM 2.1. *Let G be a finite group acting locally linearly and preserving the orientation on a Z-homology 4-sphere Σ. Then the fixed point set of G is homeomorphic to a sphere; in particular it never consists of one point.*

Local linearity is assumed to avoid pathologies, every smooth action is locally linear (see e.g. [5]).

[1]) The author has been informed that this has been proved independently by S. Cappell.