

0. Introduction

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

0. INTRODUCTION

0.1. We consider complex $n \times n$ - matrices A_1, A_2, \dots, A_s , either all unitary (case U) or all orthogonal (case O); they are called Hurwitz-Radon matrices, in short HR-matrices, if

$$(1) \quad A_j^2 = -E, \quad A_j A_k + A_k A_j = 0, \quad j, k = 1, 2, \dots, s, \quad j \neq k;$$

E or E_n denotes the unit matrix. Such matrices are well-known to exist, even with entries $0, \pm 1, \pm i$ (case U) or $0, \pm 1$ (case O). The possible values of n are multiples $mn_0, m = 1, 2, 3, \dots$ where in case $U, n_0 = 2^{s/2}$ if s is even, $n_0 = 2^{(s-1)/2}$ if s is odd. In case $O, n_0 = 2^{(s-1)/2}$ if $s \equiv 7 \pmod{8}$; $n_0 = 2^{s/2}$ if $s \equiv 0, 6$; $n_0 = 2^{(s+1)/2}$ if $s \equiv 1, 3, 5$; and $n_0 = 2^{(s+2)/2}$ if $s \equiv 2, 4 \pmod{8}$.

If we put $A_0 = E$ the relations (1) are equivalent to

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

being a unitary, or orthogonal respectively, matrix for all real x_j with $\sum_0^s x_j^2 = 1$. Thus f_s can be considered as a map $S^s \rightarrow U$ via $U(n)$, or $S^s \rightarrow O$ via $O(n)$ where $U = \varinjlim U(k)$ and $O = \varinjlim O(k)$ are the infinite unitary and orthogonal groups. We also write f_s for the homotopy class of that map, $f_s \in \pi_s(U)$ or $\pi_s(O)$. We recall that by the Bott periodicity theorems these groups are cyclic or 0.

THEOREM A. *If A_1, A_2, \dots, A_s are HR-matrices of minimal size $n = n_0(s)$ then f_s is a generator of $\pi_s(U)$, or $\pi_s(O)$ respectively, $s = 0, 1, 2, \dots$.*

Remark 1. For $s = 0$ (empty set of HR-matrices) we have $f_0(x_0) = x_0(1)$, $x_0^2 = 1$; i.e., $f_0(1) = (1)$, $f_0(-1) = (-1)$, $f_0: S^0 \rightarrow O(1) \rightarrow O$. For $s > 0$, $f_0: S^s \rightarrow O$ clearly factors through $SO(n) \rightarrow SO$ (U being connected, the analogue is irrelevant in the unitary case).

Remark 2. The problem originally solved by Hurwitz [H] and Radon [R] refers to the case O : One asks for complex bilinear forms $z = f(x, y) = \left(\sum_0^s x_j A_j\right)y$, where $z = (z_1, \dots, z_n)$, $y = (y_1, \dots, y_n)$, $x = (x_0, \dots, x_s)$, such that

$$z_1^2 + \dots + z_n^2 = (x_0^2 + \dots + x_s^2)(y_1^2 + \dots + y_n^2).$$

This means that $\sum_0^s x_j A_j$ is orthogonal, i.e. leaves invariant $\sum_0^n y_j^2$ except for the factor $\sum_0^s x_j^2$; and thus, since we may assume $A_0 = E$, that A_1, \dots, A_s is a set of orthogonal HR-matrices in the sense of (1).

The case U refers to the analogous problem for the identity

$$|z_1|^2 + \dots + |z_n|^2 = (x_0^2 + \dots + x_s^2)(|y_1|^2 + \dots + |y_n|^2)$$

where y and z are complex, and x real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination $\sum_0^s x_j A_j$ of $2n \times 2n$ -matrices with $A_0 = E$ is symplectic up to the factor $\sum_0^s x_j^2$ if and only if A_1, \dots, A_s is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group $Sp(n) \subset U(2n)$, and write Sp for the infinite symplectic group $\varinjlim Sp(k)$. With a set A_1, \dots, A_s of unitary symplectic HR-matrices, and $A_0 = E$, we associate the map $f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$, $\sum_0^s x_j^2 = 1$, of S^s into Sp via $Sp(n)$; we also write f_s for the corresponding element of $\pi_s(Sp)$, known to be 0 or cyclic.

THEOREM A'. *If A_1, \dots, A_s are unitary symplectic HR-matrices of minimal size $2n_0$ then f_s is a generator of $\pi_s(Sp)$.*

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group G_s , $s = 0, 1, 2, \dots$ introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations E_s^U and E_s^O are

computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U(E_s^O)$ turning it into a graded ring $E_*^U(E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \rightarrow \pi_s(U)$, $\psi: E_s^O \rightarrow \pi_s(O)$ given by the f_s of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from K -theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U , O and Sp .

1. THE GROUPS G_s AND THEIR REPRESENTATIONS

1.1. We will denote throughout by G_s the group given by the presentation

$$G_s = \langle \varepsilon, a_1, \dots, a_s \mid \varepsilon^2 = 1, a_j^2 = \varepsilon, a_j a_k = \varepsilon a_k a_j, j, k = 1, 2, \dots, s, j \neq k \rangle .$$

Clearly any set A_1, \dots, A_s of HR-matrices yields a (unitary or orthogonal) representation of G_s of degree n by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, \dots, s$. Conversely a representation of G_s with $\varepsilon \mapsto -E$, in short an ε -representation, yields a set of s HR-matrices. For the elementary properties of G_s and its representations we refer to [E]. We just recall that the order of G_s is 2^{s+1} , that ε is central, and that the irreducible unitary ε -representations of G_s are of degree $2^{s/2}$ if s is even (one equivalence class), of degree $2^{(s-1)/2}$ if s is odd (two equivalence classes). These degrees are the minimal values n_0 in case U . As for the case O , one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary ε -representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values n_0 (case O) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal ε -representations of G_s .

1.2. A very simple and useful scheme for studying the groups G_s and their ε -representations (and many other things) has been devised by T. Y. Lam and T. Smith [LS]. They have expressed the G_s as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; K , the Klein 4-group; and D , the dihedral group of order 8. Although K and D do not belong to the family G_s , they are of a similar nature and contain a distinguished central element ε of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the