

## §5. Type 3 case

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$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here  $F$ ,  $j_0$ , and  $Id$  are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that  $\tau(i_0) = 0$ , because  $F_*: Wh(\pi_1(U')) \rightarrow Wh(\pi_1(E(L \times I)))$  is an isomorphism. This means that  $U'$  is an  $s$ -cobordism. Therefore  $(S^{n+2}, K) \in I_0(M, L)$  by Lemma 1.6. Q.E.D.

### § 5. TYPE 3 CASE

In this section we treat the case where  $\langle m \rangle$  or  $[m]$  is of order  $p$  ( $p$  is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose  $[m]$  is of order  $p$ . Then if  $(S^{n+2}, K) \in I(M, L)$ , then  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere.*

*Proof.* Let  $r$  be the order of  $\text{Tor } H_1(M-L; \mathbf{Z})$ , and let  $\gamma$  be the canonical epimorphism  $\pi_1(M-L) \rightarrow H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$ . Since the order of  $\gamma(\langle m \rangle)$  is  $p$ , we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If  $p \geq 2$ , there are infinitely many knots  $(S^{n+2}, K)$  such that  $(S^{n+2}, K)_p$  is not a homotopy  $(n+2)$ -sphere; so Lemma 5.1 shows that  $I(M, L) \subsetneq \mathcal{K}_n$  for such  $(M, L)$ .

The rest of this section is devoted to looking for a non-trivial knot in  $I(M, L)$  or  $I_0(M, L)$ . We will extend Proposition 3.6 and 4.2 to the case where  $\langle m \rangle$  is of order  $p$ . Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let  $(S^{n+2}, K)$  be an  $n$ -knot which bounds a disk pair  $(D^{n+3}, D)$  such that  $(D^{n+3}, D)_p$  is a homotopy  $(n+3)$ -disk. Since  $(S^{n+2}, K)_p$  is the boundary of  $(D^{n+3}, D)_p$ ,  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere. If  $n+3 \geq 5$ , then  $(D^{n+3}, D)_p$  is diffeomorphic to  $D^{n+3}$  and hence  $(S^{n+2}, K)_p$  is diffeomorphic to  $S^{n+2}$ .

The  $p$ -fold branched cyclic covering  $(D^{n+3}, D)_p$  supports a  $\mathbf{Z}_p$ -action with the branch set  $D$  as the fixed point set. Let  $E(D)_p$  be the exterior of  $D$  in  $(D^{n+3}, D)_p$  and let  $\rho: S^1 \rightarrow E(D)_p$  be an equivariant embedding of a meridian of  $D$  in  $E(D)_p$ , where the standard free  $\mathbf{Z}_p$ -action is considered on  $S^1$ . Since  $\rho$  is a homology equivalence and equivariant, the Whitehead torsion of  $\rho$  is defined in  $Wh(\mathbf{Z}_p)$ . Clearly it is independent of the choice of  $\rho$ ; so we shall denote it by  $\tau_p(D^{n+3}, D)$ .

The following theorem is an extension of Proposition 3.6.

**THEOREM 5.2.** *Suppose  $\langle m \rangle$  is of order  $p$  ( $p$  may be equal to 1) for  $(M^{n+2}, L^n)$  and  $n \geq 4$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if it bounds a disk pair  $(D^{n+3}, D)$  such that*

$$(1) \quad (D^{n+3}, D)_p \text{ is diffeomorphic to } D^{n+3},$$

$$(2) \quad \mu_* \tau_p(D^{n+3}, D) = 0,$$

where  $\mu_*: Wh(\mathbf{Z}_p) \rightarrow Wh(\pi_1(M-L))$  is the homomorphism induced from a homomorphism  $\mu: \mathbf{Z}_p \rightarrow \pi_1(M-L)$  sending a generator of  $\mathbf{Z}_p$  to  $\langle m \rangle \in \pi_1(M-L)$ .

*Remark 5.3.* (1) For each  $p$ , there are infinitely many  $n$ -knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the  $\mathbf{Z}_p$ -orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact,  $\tau_p(D^{n+3}, D) = 0$  for them.

(2) If  $p = 1, 2, 3, 4$ , or  $6$ , then  $Wh(\mathbf{Z}_p) = 0$ . Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

*Proof of Theorem 5.2.* We shall observe that the proof of Proposition 3.6 works with a little modification. As before  $E(L \times I \natural D)$  can be viewed as a cobordism relative boundary between  $E(L)$  and  $E(L \# K)$ . We shall check that this is an  $s$ -cobordism.

The condition (1) implies that

$$(5.4) \quad \pi_1(E(D))/\langle m^p \rangle \simeq \mathbf{Z}_p$$

where a meridian of  $D$  in  $D^{n+3}$  is also denoted by  $m$ . Hence it follows from the decomposition (3.7) that

$$(5.5) \quad \begin{aligned} \pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{\langle m \rangle}{*} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{*} \pi_1(E(D))/\langle m^p \rangle \\ &\quad (\text{as } \langle m \rangle \text{ is of order } p \text{ in } \pi_1(E(L \times I))) \\ &\simeq \pi_1(E(L \times I)) \quad (\text{by (5.4)}) \end{aligned}$$

This implies that the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \natural D)$  induces an isomorphism  $\pi_1(E(L)) \rightarrow \pi_1(E(L \times I \natural D))$ .

We consider the map  $\tilde{i}: \tilde{E}(L) \rightarrow \tilde{E}(L \times I \natural D)$  lifted to the universal cover. Let  $q: \tilde{E}(L \times I \natural D) \rightarrow E(L \times I \natural D)$  be the covering projection map. By (5.5)  $q^{-1}(E(L \times I))$  is exactly the universal cover  $\tilde{E}(L \times I)$ . As for  $q^{-1}(E(D))$  we need a little consideration. The above observation (5.5) shows that the image of  $j_*: \pi_1(E(D)) \rightarrow \pi_1(E(L \times I \natural D))$  is isomorphic to  $\mathbf{Z}_p$ , where  $j$  is the inclusion

map. We shall identify  $j_*\pi_1(E(D))$  with  $\mathbf{Z}_p$ . Remember that  $\mathbf{Z}_p$  acts freely on  $E(D)_p$  as covering transformations.

*Claim 5.6.*  $q^{-1}(E(D)) = E(D)_p \times_{\mathbf{Z}_p} \Pi$ , where the right hand side denotes the orbit space of  $E(D)_p \times \Pi$  by the diagonal  $\mathbf{Z}_p$ -action defined by  $s \cdot (x, g) = (xs^{-1}, sg)$  for  $s \in \mathbf{Z}_p$ ,  $x \in E(D)_p$ , and  $g \in \Pi$ .

*Proof.* The  $\Pi$ -covering  $q^{-1}(E(D)) \rightarrow E(D)$  is classified by the map:  $E(D) \rightarrow B\Pi$  induced from the homomorphism  $j_*: \pi_1(E(D)) \rightarrow \Pi = \pi_1(E(L \times I \natural D))$ . Here  $j_*$  factors through the inclusion  $\ell: \mathbf{Z}_p \rightarrow \Pi$ :

$$\begin{array}{ccc} \pi_1(E(D)) & \xrightarrow{j_*} & \Pi \\ \ell \searrow & & \nearrow \ell \\ & \mathbf{Z}_p & \end{array}$$

The pullback of the universal  $\Pi$ -bundle  $E\Pi \rightarrow B\Pi$  by  $\ell$  is of the form  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\mathbf{Z}_p$ . In fact, since  $E\mathbf{Z}_p = E\Pi$ , the map  $(u, g) \rightarrow ug$  ( $u \in E\mathbf{Z}_p$ ,  $g \in \Pi$ ) is defined from  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi$  to  $E\Pi$ . The map induces a  $\Pi$ -bundle map from  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\Pi$  to  $E\Pi \rightarrow B\Pi$ . On the other hand the covering induced from the homomorphism  $\ell: \pi_1(E(D)) \rightarrow \mathbf{Z}_p$  is exactly the  $\mathbf{Z}_p$ -covering  $E(D)_p \rightarrow E(D)$ . These prove the claim.

Consequently we have a decomposition

$$(5.7) \quad \tilde{E}(L \times I \natural D) = \tilde{E}(L \times I) \cup E(D)_p \times_{\mathbf{Z}_p} \Pi,$$

where  $\tilde{E}(L \times I)$  and  $E(D)_p \times_{\mathbf{Z}_p} \Pi$  are pasted together along  $D^n \times S^1 \times_{\mathbf{Z}_p} \Pi$  equivariantly embedded in their boundaries. The condition (1) means that  $E(D)_p$  is a homology circle. This together with (5.7) tells us that  $\tilde{i}: \tilde{E}(L \times I) \rightarrow \tilde{E}(L \times I \natural D)$  induces an isomorphism on homology as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i$  is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_* \tau_p(D^{n+3}, D) \quad \text{up to sign.}$$

Hence  $\tau(i) = 0$  by the condition (2). Therefore  $E(L \times I \natural D)$  is an  $s$ -cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion  $\tau_p(S^{n+2}, K)$  is defined similarly to  $\tau_p(D^{n+3}, D)$  if  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere. The following theorem is an extension of Proposition 4.2.

THEOREM 5.8. Suppose  $\langle m \rangle$  is of order  $p$  ( $p$  may be equal to 1) for  $(M^{n+2}, L^n)$  and  $n \geq 4$ . Let  $a_{n,p} = 2$  if  $n \equiv 0 (4)$  and  $p$  is even, and let  $a_{n,p} = 1$  otherwise. Then  $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$  if

- (1)  $\sigma(S^{n+2}, K) = 0$  in case  $n$  is odd.
- (2)  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere,
- (3)  $a_{n,p} \mu_* \tau_p(S^{n+2}, K) = 0$

where  $\mu_*$  is the same as in Theorem 5.2.

*Proof.* The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

$$(5.9) \quad \begin{array}{ccc} \tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K)_p \times_{\mathbf{Z}_p} \Pi \\ \tilde{h}_1 \downarrow & & \downarrow Id \quad \downarrow h_p \times Id \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n)_p \times_{\mathbf{Z}_p} \Pi \end{array} ,$$

(see (5.7)) where  $h_p: E(K)_p \rightarrow E(S^n)_p$  denotes the lifting of  $h$  to the  $\mathbf{Z}_p$ -covers. Since  $h_p$  is a homology equivalence, the above diagram tells us that  $\tilde{h}_1$  is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = -\mu_* \tau_p(S^{n+2}, K),$$

which vanishes by the condition (3). Hence  $h_1: E(L \# K) \rightarrow E(L \# S^n)$  is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace  $\alpha$  and  $\beta$  by the canonical epimorphism  $\gamma: \mathbf{Z} \rightarrow \mathbf{Z}_p$  and  $\mu: \mathbf{Z}_p \rightarrow \Pi$  respectively. Then we have

$$\sigma(\bar{h}) = \mu_* \gamma_* \sigma(h).$$

Here we distinguish three cases to observe the value  $\sigma(\bar{h})$ .

*Case 1.* The case where  $n$  is odd. In this case the trivial homomorphism  $\alpha: \mathbf{Z} \rightarrow 1$  induces an isomorphism  $L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$  ([W11, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2,  $\alpha_*(\sigma(h))$  vanishes. Hence  $\sigma(h) = 0$ , so  $\sigma(\bar{h}) = 0$ .

*Case 2.* The case where  $n \equiv 2 (4)$  or  $p$  is odd. According to Wall [W12] or Bak [Ba],  $L_{n+3}(\mathbf{Z}_p, 1) = 0$  in this case. Since  $\gamma_* \sigma(h)$  lies in  $L_{n+3}(\mathbf{Z}_p, 1)$ ,  $\gamma_* \sigma(h) = 0$  and hence  $\sigma(\bar{h}) = 0$ .

Case 3. The case where  $n \equiv 0(4)$  and  $p$  is even. In this case  $L_{n+3}(\mathbf{Z}_p, 1) \simeq \mathbf{Z}_2$ . Since the value  $\gamma_*\sigma(h) \in L_{n+3}(\mathbf{Z}_p, 1)$  is additive with respect to connected sum, it necessarily vanishes for  $(S^{n+2}, K) \# (S^{n+2}, K)$ .

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

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