4. Relative de Rham homology

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If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^{\bullet}(X, \mathbf{C}) \xrightarrow{\sim} H^{\bullet}_{\infty}(X, \mathbf{C})$$

from de Rham cohomology to smooth singular cohomology.

4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact p-chain T on a smooth n-dimensional manifold X. Since we can realize T as a section in the sheaf $\Omega^{p\vee}$ the general sheaf theoretic notion of support applies: The *support* of T, $\operatorname{Supp}(T)$ is the smallest closed subset Z of X, such that the restriction of T to $X - Z$ is zero.

(4.1) EXAMPLE. Integration over an oriented compact p-dimensional submanifold K with boundary defines a compact p-chain κ with $\operatorname{Supp}(\kappa) = K$. From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega\,, \quad \omega \in \Gamma(X, \Omega^p)\,,$$

we conclude that $\operatorname{Supp}(b\kappa) = \partial K$.

Let us now consider the inclusion $j\colon U \to X$ of an open subset U of X. The induced map

$$j_*\colon D^c_p(U, \mathbf{C}) \to D^c_p(X, \mathbf{C})\,, \quad p \in \mathbf{N}\,,$$

is injective since we may interpret j_* as "extension by zero" in the sheaf $\Omega_p^{\vee}$, compare (2.5). A compact p-chain T on X belongs to the image of j_* if and only if $\operatorname{Supp}(T) \subseteq U$. The complex $D^c_{\bullet}(X, U; \mathbf{C})$ of *relative compact chains* is defined to fit into the following exact sequence

$$0 \to D^c_{\bullet}(U, \mathbf{C}) \xrightarrow{j_*} D^c_{\bullet}(X, \mathbf{C}) \to D^c_{\bullet}(X, U; \mathbf{C}) \to 0\,. \tag{4.2}$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D^c_{\bullet}(X, U; \mathbf{C})\,, \quad p \in \mathbf{N}\,.$$

In more concrete terms we can describe this homology group as

$$(4.3)\qquad \{Z \in D^c_p(X, \mathbf{C}) \mid \operatorname{Supp}(bZ) \subseteq U\} \Big/ \{bW \mid W \in D^c_{p+1}(X, \mathbf{C})\} + \{Z \in D_p(X, \mathbf{C}) \mid \operatorname{Supp}(Z) \subseteq U\}$$

From the exact sequence (4.2) we deduce the homology sequence

$$(4.4)\qquad \to H^c_p(U, \mathbf{C}) \to H^c_p(X, \mathbf{C}) \to H^c_p(X, U; \mathbf{C}) \to H^c_{p-1}(U, \mathbf{C}) \to H^c_{p-1}(X, \mathbf{C}) \to$$

Let $f: X \to Y$ denote a smooth map, U an open subset of X and V an open subset of Y containing $f(U)$. Let us notice that

$$(4.5)\qquad \operatorname{Supp}(f_* T) \subseteq f(\operatorname{Supp}(T)), \quad T \in D^c_p(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let Z be a closed subset of X and Y an open subset of X containing Z. The inclusion of $V = Y - Z$ in $U = X - Z$ induces an isomorphism

$$H^c_\bullet(Y, V; \mathbf{C}) \tilde{\to} H^c_\bullet(X, U; \mathbf{C}).$$

Proof. Let $i: Z \to X$ denote the inclusion. From the fact that $\Omega^{\bullet\vee}$ consists of soft sheaves we deduce an exact sequence

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \to \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, i^*\Omega^{\bullet\vee}) \to 0$$

compare [5] III. 7.6. This alows us to make the identification

$$(4.7)\qquad D^c_\bullet(X, U; \mathbf{C}) \tilde{\to} \Gamma_c(Z, i^*\Omega^{\bullet\vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when X is replaced by Y and U by V. Q.E.D.

(4.8) *Continuity.* Let (X_α) be an outward directed open covering of the manifold X. For any open subset U of X we have that

$$\lim_{\to\alpha} H^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H^c_\bullet(X, U; \mathbf{C})$$

Proof. As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\to} D^c_\bullet(X_\alpha, \mathbf{C}) = D^c_\bullet(X, \mathbf{C})$$

and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D_\bullet^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D_\bullet^c(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

(4.9) $$\bigoplus_\alpha H_\bullet^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H_\bullet^c(X, U; \mathbf{C}).$$

5. Stokes formula

Let us consider the open subset U of the n-dimensional smooth manifold X and the resulting exact sequences

(5.1) $$\begin{array}{l} \to H_p^c(X, \mathbf{C}) \to H_p^c(X, U; \mathbf{C}) \xrightarrow{b} H_{p-1}^c(U, \mathbf{C}) \xrightarrow{j_*} H_{p-1}^c(X, \mathbf{C}) \to \\ \leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{array}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

(5.2) $$H_Z^p(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) Theorem. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for $\alpha \in H_p^c(X, U; \mathbf{C})$ *and* $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, \Omega^{\bullet\vee}) \to 0.$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet\vee\vee}$ of $\mathbf{R}$ established in the proof of the biduality theorem (2.1).