6. Coordinate free tensor calculus

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5. A priori estimates: the original way

According to proposition 4.3 we must prove now that, given any sequence of positive real numbers (K_i), $i \in \mathbf{N}$, there exists a sequence (C_i) such that

$$\forall i \in \mathbf{N}\,, \quad \| D^i P_\lambda(\varphi) \| \leqslant K_i$$

implies

$$\| \varphi \| \leqslant C_0\,, \quad \forall i \in \mathbf{N}\,, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leqslant C_{i+2}\,.$$

These are *a priori* estimates of order zero, two, three, and so on .... In case $\lambda > 0$, the C^0 estimate is straightforward [2]. In case $\lambda = 0$, it becomes very tricky; proofs simpler than Yau's original one [24] (p. 352-359), based on the idea of uniformly estimating the $L^p(dX_g)$ norms of φ, may be found in [16] (dimension 2), [3] [21] and [4] (p. 148-149).

Estimates of order two and three are carried out by means of tensor calculus and of the Maximum Principle (for elliptic equations) [20] applied to *suitable* test functions. Though it is not everywhere clear in [21] [24], it is worth noting that the computations can be written intrinsically, i.e. without any reference to a *particular* system of coordinates (e.g. [2]), or even *coordinate free* (see section 6 below).

Further regularity is then recovered by Schauder theory e.g. [5] (lemma 1). In the sequel, we show how further estimates can be carried out instead, *just going ahead with coordinate free tensor calculus.* This occurs actually for any fully nonlinear second order elliptic equation on a compact Riemannian manifold, via a straightforward imitation of the device below.

Remark 5.1. It follows from the C^2 *a priori* estimates that the metrics g' are *a priori* uniformly equivalent to the original metric g (see e.g. [3], p. 75).

6. Coordinate free tensor calculus

Even coordinate free tensor calculus needs indices. Usually these indices refer to a *local* frame. Another way is to view these indices *globally* as labelling copies of the holomorphic and antiholomorphic tangent and cotangent bundles. From this point of view, a tensor written with indices is a section of the tensor product of a family of bundles indexed by an *unordered* set of indices (disregarding those indices subject to the summation convention).

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \quad \text{while}$$
$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots} \quad .$$

As usual if $T_{a \dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$T_{a \dots lm} \text{ stands for } \nabla_m T_{a \dots l}, \quad \text{while}$$
$$T_{a \dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a \dots l} \quad .$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = a\bar{b}c$, then $\bar{\alpha} = \bar{a}b\bar{c}$), while $| \alpha |$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations D, ∇, $\bar{\nabla}$, $\| \ \|$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\varphi$)

$$\varphi_{aa'\alpha b} - (\varphi_{aa'\alpha})_b \equiv \varphi_{ac\alpha}\, \varphi_{a'c'b} \tag{3}$$

does not vanish.

7. Higher order a priori estimates: generalities

We want to prove by induction,

Proposition 7.1. *Given* $n \geqslant 4$, *a sequence* (K_i), $i \in \mathbf{N}$, *and a finite sequence* $C_0, \dots, C_{n-1}$, *there exists* C_n *such that:*

$$\| \varphi \| \leqslant C_0 , \quad \forall i = 0, \dots, n-3 , \quad \| D^i \nabla\bar{\nabla}\varphi \| \leqslant C_{i+2}$$
$$\text{and} \quad \forall i \in \mathbf{N} , \quad \| D^i P_\lambda(\varphi) \| \leqslant K_i ,$$

implies

$$\| D^{n-2} \nabla\bar{\nabla}\varphi \| \leqslant C_n .$$