

# 6. Coordinate free tensor calculus

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## 5. A PRIORI ESTIMATES: THE ORIGINAL WAY

According to proposition 4.3 we must prove now that, given any sequence of positive real numbers  $(K_i), i \in \mathbf{N}$ , there exists a sequence  $(C_i)$  such that

$$\forall i \in \mathbf{N}, \quad \| D^i P_\lambda(\varphi) \| \leq K_i$$

implies

$$\| \varphi \| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

These are *a priori* estimates of order zero, two, three, and so on ... In case  $\lambda > 0$ , the  $C^0$  estimate is straightforward [2]. In case  $\lambda = 0$ , it becomes very tricky; proofs simpler than Yau's original one [24] (p. 352-359), based on the idea of uniformly estimating the  $L^p(dX_g)$  norms of  $\varphi$ , may be found in [16] (dimension 2), [3] [21] and [4] (p. 148-149).

Estimates of order two and three are carried out by means of tensor calculus and of the Maximum Principle (for elliptic equations) [20] applied to *suitable* test functions. Though it is not everywhere clear in [21] [24], it is worth noting that the computations can be written intrinsically, i.e. without any reference to a *particular* system of coordinates (e.g. [2]), or even *coordinate free* (see section 6 below).

Further regularity is then recovered by Schauder theory e.g. [5] (lemma 1). In the sequel, we show how further estimates can be carried out instead, *just going ahead with coordinate free tensor calculus*. This occurs actually for any fully nonlinear second order elliptic equation on a compact Riemannian manifold, via a straightforward imitation of the device below.

*Remark 5.1.* It follows from the  $C^2$  *a priori* estimates that the metrics  $g'$  are *a priori* uniformly equivalent to the original metric  $g$  (see e.g. [3], p. 75).

## 6. COORDINATE FREE TENSOR CALCULUS

Even coordinate free tensor calculus needs indices. Usually these indices refer to a *local* frame. Another way is to view these indices *globally* as labelling copies of the holomorphic and antiholomorphic tangent and cotangent bundles. From this point of view, a tensor written with indices is a section of the tensor product of a family of bundles indexed by an *unordered* set of indices (disregarding those indices subject to the summation convention).

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to  $g$  or to  $g'$  according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \text{ while}$$

$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots} .$$

As usual if  $T_{a\dots l}$  is a tensor, further lower indices refer to covariant differentiation (with respect to  $g$ ); so,

$$T_{a\dots lm} \text{ stands for } \nabla_m T_{a\dots l}, \text{ while}$$

$$T_{a\dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a\dots l} .$$

Our indices will be latin letters; greek letters will denote multi-indices. If  $\alpha$  is a multi-index,  $\bar{\alpha}$  will denote the *conjugate* multi-index (for instance if  $\alpha = \bar{a}\bar{b}\bar{c}$ , then  $\bar{\alpha} = \bar{a}\bar{b}\bar{c}$ ), while  $|\alpha|$  denotes its length. We shall say that  $\alpha$  is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations  $D, \nabla, \bar{\nabla}, \| \cdot \|$ , were introduced in section 4.

*Remark 6.1.* Since covariant differentiation (with respect to  $g$ ) and contraction with respect to  $g'$  *do not* commute, we observe that, for instance, the difference (recall  $g' = g + \nabla\bar{\nabla}\phi$ )

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

## 7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given  $n \geq 4$ , a sequence  $(K_i), i \in \mathbf{N}$ , and a finite sequence  $C_0, \dots, C_{n-1}$ , there exists  $C_n$  such that:*

$$\| \phi \| \leq C_0, \quad \forall i = 0, \dots, n - 3, \quad \| D^i \nabla \bar{\nabla} \phi \| \leq C_{i+2}$$

and  $\forall i \in \mathbf{N}, \quad \| D^i P_\lambda(\phi) \| \leq K_i,$

*implies*

$$\| D^{n-2} \nabla \bar{\nabla} \phi \| \leq C_n .$$