2. HOMOGENEOUS CONTACT MANIFOLDS

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2. Homogeneous contact manifolds

We formulate the notion of contact manifold in terms of complex analytic manifolds; the definitions apply equally to real smooth manifolds. Especially, if the complex analytic manifold is a smooth algebraic variety defined over $\mathbf{R}$ and if its various structures are defined over $\mathbf{R}$, then the elementary assertions here apply to the set of real points of the variety.

Throughout this section we indicate proofs only when they differ from those of Boothby [1, 2] or Wolf [7, 8].

2.1. A *complex contact manifold* is a complex manifold M of odd dimension $2n-1$ together with a *complex contact structure* which is prescribed by a family $\{ (U_\alpha, \omega_\alpha) \}$ consisting of an open cover $\{ U_\alpha \}$ of M and holomorphic Pfaffian forms ω_α on U_α satisfying:

(i) $\omega_\alpha \wedge (d\omega_\alpha)^{n-1}$ does not vanish on U_α, i.e., ω_α is of maximal rank;

(ii) if $U_\alpha \cap U_\beta$ is not empty, then $\omega_\beta = f_{\beta\alpha}\omega_\alpha$ on $U_\alpha \cap U_\beta$ with $f_{\beta\alpha}$ holomorphic and non-vanishing; and

(iii) the family $\{ (U_\alpha, \omega_\alpha) \}$ is maximal with respect to (i) and (ii).

A holomorphic map $g\colon M \to M'$ between two contact manifolds is a *contact transformation* if $\{ (g^{-1}U'_\alpha, g^* \omega'_\alpha) \}$ is contained in $\{ (U_\alpha, \omega_\alpha) \}$. [1, §2].

2.2. Let M be the space of hypersurface elements in complex Euclidean space E^n whose hyperplanes meet the x_n-axis, that is, points $(x_1, \ldots, x_n)$ and incident hyperplanes

$$x'_n - x_n = p_1 (x'_1 - x_1) + \ldots + p_{n-1} (x'_{n-1} - x_{n-1}),$$

where primes denote running coordinates. The single set of coordinates $x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$ on M together with the Pfaffian form

$$\omega = dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1}$$

suffices to define a contact structure on M [6, §63]. This is *the* classical contact manifold to which we will relate all others.

2.3. The contact structure on a complex manifold M has been formulated by S. Kobayashi in terms of a principal $\mathbf{C}^*$-bundle over M [1, §2 and 7, §2]. Let $\{ (U_\alpha, \omega_\alpha) \}$ be a contact structure on M, so that $\omega_\beta = f_{\beta\alpha}\omega_\alpha$ on $U_\alpha \cap U_\beta$. Define the holomorphic principal $\mathbf{C}^*$-bundle $\pi\colon B \to M$ using the transition functions $f_{\beta\alpha}{}^{-1} = \dfrac{1}{f_{\beta\alpha}}$ on $U_\alpha \cap U_\beta$; $\pi^{-1}(U_\alpha)$ is $U_\alpha \times \mathbf{C}^*$ and, with coordinate z_α on $\mathbf{C}^*$, $z_\beta = f_{\beta\alpha}{}^{-1} z_\alpha$ on $U_\alpha \cap U_\beta$. ω_α on U_α pulls back by π^* to a Pfaffian form on $\pi^{-1}(U_\alpha)$, again denoted ω_α. On $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ we have $\omega_\beta = f_{\beta\alpha}\,\omega_\alpha$ and $z_\beta = f_{\beta\alpha}{}^{-1} z_\alpha$ so that $z_\alpha\omega_\alpha = z_\beta\omega_\beta$; the $z_\alpha\omega_\alpha$ hence define a holomorphic Pfaffian form ω on B. Let R_a, a in $\mathbf{C}^*$, denote the right action of $\mathbf{C}^*$ on B. The Pfaffian form ω satisfies:

(a) $(d\omega)^n$ does not vanish on B;

(b) ω vanishes on vectors tangent to the fibers of B; and

(c) $R_a{}^*\omega = a\omega$, a in $\mathbf{C}^*$ [1, (2.1)].

Conversely, a holomorphic principal $\mathbf{C}^*$-bundle B over M together with a holomorphic Pfaffian form ω satisfying (a, b, c) defines a contact structure on M. The ω_α on U_α are obtained by pulling down ω by sections of B over U_α.

Complex contact structures on M correspond uniquely to principal $\mathbf{C}^*$-bundles $\pi\colon B \to M$ equipped with a Pfaffian form ω satisfying (a, b, c), up to isomorphism [1, (2.1)]. Contact transformations $M \to M'$ are exactly those homomorphisms $g\colon B \to B'$, $\pi' \circ g = g \circ \pi$ and $R'_a \circ g = g \circ R_a$, satisfying $g^*\omega' = \omega$. Consequently, contact automorphisms of M correspond to bundle automorphisms g of B which preserve ω: $g^*\omega = \omega$ [1, (3.1)].

In case M is *compact*, its group of all contact automorphisms is a *complex Lie group* which acts holomorphically on B [1, (3.2) and 2, §1].

2.4. Let V be a complex manifold of dimension n and M the bundle of complex co-directions of V, that is, M is obtained from the bundle B which is the cotangent bundle of V less its zero section by passing to the projective space of each fiber. B is a principal $\mathbf{C}^*$-bundle over M. If $x_1, \dots, x_n$ are coordinates on an open set U of V, ξ in B may be written over U as

$$\xi = u_1(\xi)\, dx_1 + \dots + u_n(\xi)\, dx_n\,,$$

where the functions $u_i(\xi)$ are homogeneous of degree one; the functions $x_1, \dots, x_n$, $u_1, \dots, u_n$ define coordinates in B over U. The Pfaffian form

$$\omega = u_1 dx_1 + \dots + u_n dx_n$$

on B satisfies (a, b, c) of 2.3 and hence defines a contact structure on M. Again, this is classical [6, §63, p. 242]. We may cover M over U by open sets where some u_i is not zero, say $u_n \neq 0$, and then set

$$u_1 = -p_1, \ldots, u_{n-1} = -p_{n-1}, u_n = 1;$$

this gives a section of B over this open set and ω pulls down to

$$dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1},$$

the description in 2.2.

2.5. Let V of 2.4 be complex projective space P^n. Points of P^n are described by homogeneous coordinates $x_0, \ldots, x_n$, written as a column vector $x = {}^t(x_1, \ldots, x_n)$, and hyperplanes

$$ux' = u_0 x_0' + \ldots + u_n x_n' = 0$$

of P^n by homogeneous coordinates $u_0, \ldots, u_n$, written as a row vector $u = (u_0, \ldots, u_n)$. A cotangent vector at x is determined by the equation of a hyperplane u incident with x, $ux = 0$; if x is replaced by λx, u must be replaced by $u\lambda^{-1}$. Thus B may be described by (x, u), $ux = 0$, with $(\lambda x, u\lambda^{-1})$ equivalent to (x, u). M is consequently described by incident points and hyperplanes (x, u), $ux = 0$; now $(\lambda x, u\mu^{-1})$ is equivalent to (x, u). The Pfaffian form

$$\omega = udx = u_0 dx_0 + \ldots + u_n dx_n$$

is well-defined on B and gives the contact structure on the space M of co-directions, that is, on the space of hypersurface elements, of P^n [6, §63, p. 242].

A projectivity of P^n, a transformation in $PSL(n+1; \mathbf{C})$, which will be represented by $x \to gx$ with g in $SL(n+1; \mathbf{C})$, transforms hyperplanes by $u \to ug^{-1}$ and cotangent vectors and co-directions by $(x, u) \to (gx, ug^{-1})$. Since

$$g^*\omega = g^*(udx) = ug^{-1} d(gx) = ug^{-1} g dx = udx = \omega,$$

projectivities are contact transformations of M. In addition, since $ux = 0$, $udx = -{}^t x {}^t du$ and, hence, classical projective duality $(x, u) \to ({}^t u, -{}^t x)$ preserves ω and is a contact transformation [6, §62].

2.6. Let M be a complex contact manifold which is *algebraic*; M is a smooth algebraic variety and the contact structure is given by a bundle $B \to M$ and Pfaffian form ω on B which are algebraic.

Assume further that M is *connected, compact,* and *homogeneous* under a *linear algebraic group* G of contact automorphisms. Since M is connected, we may assume G is *connected.* We may also assume that G acts *effectively* on M: only the identity element of G acts as the identity transformation on M.

Now G is *semi-simple* [1, §4]. For the radical of G, a normal solvable subgroup, has a fixed point in compact M [3, (10.4)] and since, G acts effectively on M, this radical is trivial. Thus M is exhibited as G/P, with G connected and semi-simple, and P the isotropy subgroup of a point x_0 in M. Since G/P is compact, P is a *parabolic* subgroup of G [3, (11.2); 4, §68 ff.; and 8, §2]. P is its own normalizer in G, so contains the center of G; since G acts effectively on M, this center is trivial. G is *centerless.*

Since G is a linear algebraic group, we will throughout view the elements of it and its Lie algebra $\mathfrak{g}$ as matrices. Thus: For g in G and X in $\mathfrak{g}$, $Ad\,(g)\,X = gXg^{-1}$, a product of matrices. Left-invariant Pfaffian forms on G are given by $\omega_0\,(g^{-1}dg)$, where ω_0 is a linear function on $\mathfrak{g}$, and dg is the matrix of differentials of the entries of g. The action of $Ad\,(g)$ on left-invariant Pfaffian forms on G, i.e., $Ad\,(g)^*$, is then $({}^tAd(g)\omega_0)\,(X) = \omega_0\,(gXg^{-1})$.

2.7. From 2.3, G acts on the principal $\mathbf{C}^*$-bundle B over $M = G/P$. Let b_0 in B lie over the point x_0 in M fixed by P. If g is in P, then gb_0 lies over x_0, so $gb_0 = R_a b_0 = b_0 a$ for a unique $a = \chi\,(g)$ in $\mathbf{C}^*$. $\chi\colon P \to \mathbf{C}^*$ is a homomorphism. χ is either surjective or trivial, and in the former case G is transitive on B since it is then transitive on M and on the fibers of B over M. In fact, χ *is* surjective [2, §2]; the key lemma of Boothby's argument [2, p. 277] may be replaced by: The centralizer in $\mathfrak{g}$ of a nonzero element of $\mathfrak{g}$ is never a parabolic subalgebra. Thus B is exhibited as G/P_1 with P_1, the kernel of χ, a subgroup of P. The bundle $B \to M$ is $G/P_1 \to G/P$ with fiber $P/P_1 \simeq \mathbf{C}^*$.

By means of the map $G \to G/P_1$, pull the Pfaffian form ω on $B = G/P$, which defines the contact structure, up to a left-invariant form $\omega_0\,(g^{-1}dg)$ on G. This form is *$Ad\,(P_1)$-invariant*:

$$\omega_0\,(gXg^{-1}) = \omega_0\,(X)\,, \ g \text{ in } P_1\,, \ X \text{ in } \mathfrak{g}\,.$$

Let $\mathfrak{p}$ and $\mathfrak{p}_1$ denote the Lie algebras of P and P_1, respectively. Conditions (a, b, c) of 2.3 become:

(a) $(d\omega_0)^n \neq 0$;

(b) $\omega_0(X) = 0$, X in $\mathfrak{p}$; and

(c) $\omega_0(g^{-1}Xg) = \chi(g)\,\omega_0(X)$, g in P, X in $\mathfrak{g}$;

where $d\omega_0(X, Y) = -\frac{1}{2}\omega_0([X, Y])$ [1, (5.1), (5.2), (5.3)].

Since $\mathfrak{g}$ is semi-simple, its Killing form is non-degenerate and we may write

$$\omega_0(X) = \langle W, X \rangle, \; X \text{ in } \mathfrak{g},$$

for a unique W in $\mathfrak{g}$. Conditions (a, b, c) now become:

(a′) the centralizer of W in $\mathfrak{g}$ is $\mathfrak{p}_1$;

(b′) $\langle W, X \rangle = 0$, X in $\mathfrak{p}$; and

(c′) $[X, W] = \chi'(X)\,W$, X in $\mathfrak{p}$;

where χ' is the derivative of χ at the identity of P [1, (5.6)].

As a consequence of (c′), $\rho = \chi'$ restricted to a Cartan subalgebra contained in $\mathfrak{p}$ is a root of $\mathfrak{g}$; $E_\rho = W$ may be taken as the corresponding root vector. When the roots of $\mathfrak{g}$ are ordered, ρ is a positive root and $\rho + \alpha$, α a positive root, is not a root [1, (6.2)]. Hence ρ is the *maximal* root for this ordering and G is *simple* [1, (6.3) and 4, (25.6)].

2.8 Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, and choose a system of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Designate a subset of the simple roots as free and call the remaining simple roots non-free. An arbitrary root is called *free* if it contains a free simple root as a summand, and *non-free* if all its summands are non-free simple roots. A free root is necessarily positive. Besides free and non-free roots there are only the negatives of free roots [4, (69.23)]. If $\mathfrak{g}_\alpha$ denotes the root space for the root α, then

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \text{ non-free}} \mathfrak{g}_\alpha + \sum_{\alpha \text{ free}} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$, that is, it corresponds to a parabolic subgroup of any connected complex Lie group G having $\mathfrak{g}$ as its Lie algebra. Now, any parabolic subalgebra of $\mathfrak{g}$ is $Ad(G)$-conjugate to a parabolic subalgebra given by the above construction. Thus, once $\mathfrak{h}$ and the system of simple roots are fixed, the subsets of the simple roots classify parabolic subalgebras up to conjugacy [8, §2].

2.9 Continuing 2.7: We may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{p}$, choose a system of simple roots, and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8 [8, §2]. The free and non-free roots are completely determined by the maximal root ρ, in that

$$\langle H_\rho, H_\alpha \rangle > 0 \text{ for } \alpha \text{ free,}$$
$$\langle H_\rho, H_\alpha \rangle = 0 \text{ for } \alpha \text{ non-free,}$$

where H_α in $\mathfrak{h}$ is defined by $\langle H_\alpha, H \rangle = \alpha(H)$, H in $\mathfrak{h}$ [1, (6.5)]. Consequently, we may describe $\mathfrak{p}, \mathfrak{p}_1$, and ω_0 for an algebraic homogeneous contact manifold in terms of the maximal root ρ by

(i) $\mathfrak{p} = \mathfrak{h} + \sum_{\langle H_\rho, H_\alpha \rangle \geqq 0} \mathfrak{g}_\alpha$,

(ii) $\mathfrak{p}_1 =$ elements X of $\mathfrak{p}$ orthogonal to H_ρ, $\langle H_\rho, X \rangle = 0$, and

(iii) $\omega_0(X) = \langle E_\rho, X \rangle$, X in $\mathfrak{g}$,

[7, p. 1035]. Since G is connected and centerless with Lie algebra $\mathfrak{g}$, the groups G, P, P_1 and the form ω are completely determined.

2.10. Conversely, begin with a simple complex Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots. Using the maximal root ρ, define $\mathfrak{p}$, $\mathfrak{p}_1$, and ω_0 as in (i, ii, iii) of 2.9. Take for G the adjoint group of $\mathfrak{g}$, which is connected, centerless, and simple, and for P and P_1 the subgroups corresponding to $\mathfrak{p}$ and $\mathfrak{p}_1$. Then, $\omega_0(g^{-1}\,dg)$ is a left-invariant, $Ad(P_1)$-invariant Pfaffian form on G, and defines a form ω on G/P_1. The map $X \to \langle H_\rho, X \rangle$, X in $\mathfrak{p}$, gives rise to a homomorphism $\chi\colon P \to \mathbf{C}^*$. The form ω on the principal $\mathbf{C}^*$-bundle G/P_1 over G/P satisfies (a, b, c) of 2.3 and hence defines a contact structure making G/P a compact homogeneous algebraic contact manifold [1, Th. C and 7, p. 1035].

In this manner, Boothby established that there is exactly one compact homogeneous algebraic contact manifold, up to isomorphism, for each type $A_n, \ldots, G_2$ of simple complex Lie algebra [1, (7.1)]. For these manifolds, the group G is the identity component of the group of all contact automorphisms [7, (2.5)]. Boothby's classification [1, 2] was obtained with the assumptions that the complex contact manifold was compact, simply connected, and homogeneous under a complex Lie group of contact transformations, and used H. C. Wang's theory of compact homogeneous complex manifolds rather than parabolic subgroups. We may conclude that these contact manifolds are algebraic.

2.11. Let G be a semi-simple complex Lie group and G_0 a real form of G: G_0 is the set of elements of G fixed under a complex conjugation $g \to \bar{g}$. We use a bar to denote the conjugate of an object, and the terms *real* and *stable* refer to the conjugation.

Let P be a parabolic subgroup of G, and $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{p}$ the Lie algebras of G, G_0, P. $\mathfrak{g}_0 \cap \mathfrak{p}$ contains a stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\bar{\mathfrak{h}} = \mathfrak{h}$ [8, (2.6)]. If α is a root of $\mathfrak{g}$, so is $\bar{\alpha}$; α is real if $\bar{\alpha} = \alpha$. Now, choose a system of simple roots and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8. Set $P_0 = G_0 \cap P$; G_0/P_0 is a subset of G/P. Wolf has shown that the following are *equivalent:*

(i) real dimension of G_0/P_0 = complex dimension of G/P,

(ii) G_0/P_0 is closed in G/P,

(iii) the set of free roots is stable,

(iv) $\mathfrak{p}$ or P is stable, and

(v) the algebraic manifold G/P is defined over **R** and G_0/P_0 is its set of real points,

[8, (3.6)]. When these conditions hold, G_0/P_0 is the unique closed orbit of G_0 on G/P [8, (3.4)]. We call G_0/P_0 a *real form* of G/P.

Let M be a compact, algebraic homogeneous contact manifold. Assume that M and its contact structure are *defined over* **R**, that is, the principal $\mathbf{C}^*$-bundle $B \to M$ and the Pfaffian form ω on B are defined over **R**. Let P be the isotropy subgroup of a real point x_0 of M in the group G of contact automorphisms. Then the complex conjugation on M defines one on G, $\bar{g}x = \overline{g\bar{x}}$, under which P is stable. Hence, we obtain a real form G_0 of G so that the real points of M are G_0/P_0, $P_0 = G_0 \cap P$. That ω is defined over **R** means $W = E_\rho$ lines in $\mathfrak{g}_0$, and the maximal root ρ is *real.* This is consistent with the stability of the set of free roots, as $\langle H_\rho, H_{\bar{\alpha}} \rangle > 0$ when $\langle H_\rho, H_\alpha \rangle > 0$. Consequently, the real forms of M correspond to the conjugations of $\mathfrak{g}$ for which ρ is real.

2.12. The method by which the contact structure on G/P will be exhibited, in the next sections, in classical form 2.2 is the following.

Let

$$\mathfrak{m} = \sum_{\langle H_\rho, H_\alpha \rangle < 0} \mathfrak{g}_\alpha ;$$

$\mathfrak{m}$ is supplementary to $\mathfrak{p}$ in $\mathfrak{g}$ and of dimension $2n-1$. We will determine X near 0 in $\mathfrak{m}$ as a function of $x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$ so that $X \to (\exp X) \cdot x_0$

is one-to-one on an open neighborhood U of x_0 in G/P and $(\exp X) \cdot x_0$ is identifiable as the point $(x_1, ..., x_n)$ and the incident hyperplane

$$x_n' - x_n = p_1(x_1' - x_1) + ... + p_{n-1}(x_{n-1}' - x_{n-1}).$$

Now, $(\exp X) \cdot x_0 \to (\exp X) \cdot b_0$ is a section of the bundle G/P_1 over U and, via this section, the form ω on G/P_1 pulls down to

$$\omega_0((\exp X)^{-1} d(\exp X))$$

which, when expressed in terms of $x_1, ..., x_n, p_1, ..., p_{n-1}$, will be identified with

$$dx_n - p_1 dx_1 - ... - p_{n-1}dx_{n-1}$$

up to a constant multiple $a \neq 0$. For this latter calculation we will use

$$(\exp X)^{-1} d(\exp X) = \frac{1 - e^{-ad\,X}}{ad\,X}(dX)$$
$$= dX - \frac{1}{2}[X, dX] + \frac{1}{6}[X, [X, dX]] - ...$$

[4, (10.2)], a series which is finite since $\mathfrak{m}$ is nilpotent. In fact, our choice of X will make the series for $\exp X$ themselves finite. The constant $a \neq 0$ could be made unity by using instead the section $(\exp X) \cdot x_0 \to (\exp X) g^{-1} \cdot b_0$, where g in P is chosen so that $\chi(g) = a$. This amounts to following the original section by R_a^{-1} in the bundle.

3. Co-directions in projective space

The contact structure on the $(2n-1)$-dimensional space of co-directions in complex projective space P^n, described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type A_n, $n \geqslant 1$.

3.1 Let $\mathfrak{g} = \mathfrak{sl}(n+1; \mathbf{C})$, complex $(n+1)$ by $(n+1)$ matrices of trace zero. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take the diagonal matrices of $\mathfrak{g}$. Let δ_i, $i = 0, 1, ..., n$ be the linear function on $\mathfrak{h}$ which assigns to $H = \operatorname{diag}(h_1, ..., h_n)$ in $\mathfrak{h}$ the i^{th} diagonal element: $\delta_i(H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\delta_i - \delta_j \quad i, j = 0, 1, ..., n$$
$$\text{and } i \neq j$$