IV. Recent progress on uniqueness problems

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It is a well-known fact that even for absolutely minimizing surfaces the minimum need not be unique. It is expected that in case there are two or more absolute minima, a small deformation of the boundary will separate them, restoring uniqueness of absolute minimum. This has been recently done by F. Morgan [M], who proves that almost every C^3 closed curve in $\mathbf{R}^3$ bounds a unique minimal surface of least area. It would be however too technical to describe this result in more detail and, going to the opposite point of view, I will give an explicit example of a 2 dimensional compact manifold in $\mathbf{R}^4$ bounding infinitely many oriented stationary manifolds of dimension 3. In fact, our example will be the Clifford flat torus

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2$$

which is also minimal in S^3.

Theorem 2. *The Clifford flat torus in* $\mathbf{R}^4$ *bounds infinitely many 3-dimensional manifolds with mean curvature* 0 *at every point.*

We sketch the proof of this result, which is implicit in the paper [B-DG-G] on minimal cones and the Bernstein problem.

Let $p + q = n - 2$ and consider the action of $SO(p) \times SO(q)$ on $\mathbf{R}^n = \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$. Let $u = (x_1^2 + \ldots + x_{p+1}^2)^{1/2}$, $v = (x_{p+2}^2 + \ldots + x_{p+q+2}^2)^{1/2}$. If we consider minimal hypersurfaces in $\mathbf{R}^n$ invariant by $SO(p) \times SO(q)$, we may describe them in the form $f(u, v) = 0$ with u, v as above, and thus as a curve Γ in the quadrant $u, v \geqq 0$. If we now represent Γ parametrically as $(u(t), v(t))$ the condition of mean curvature 0 on the hypersurface means that

$$u'' v' - u' v'' + [p(u')^2 + q(v')^2]\left(\frac{u'}{v} - \frac{v'}{u}\right) = 0$$

or in other words that Γ is a geodesic for the metric

$$ds^2 = u^p v^q [(du)^2 + (dv)^2] .$$

In our case

$$ds^2 = (uv)[(du)^2 + (dv)^2]$$

and the requirement that our hypersurface has the Clifford torus as boundary means that we have to find all geodesics which start at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and end at $uv = 0$. There is exactly one such geodesic ending at (0, 0), namely

$u = v$. This corresponds to the well-known cone $x_1^2 + x_2^2 = x_3^2 + x_4^2$, which has indeed mean curvature 0. The other possibility for a geodesic is to end on the u-axis, those ending on the v-axis being obtained by a symmetrical reflection. Up to a homothetic transformation there is only one such geodesic. We introduce the new homothetically invariant parameters

$$\varphi = \operatorname{artg} \frac{v}{u}, \quad \theta = \operatorname{artg} \frac{v'}{u'},$$

$$\sigma = \theta - 3\varphi + \frac{\pi}{2}, \quad \psi = \theta + \varphi - \frac{\pi}{2}$$

and rewrite the equation for geodesics as

$$\begin{cases} \dot{\sigma} = -\dfrac{3}{2} \sin \sigma - \dfrac{7}{2} \sin \psi \\[2ex] \dot{\psi} = \dfrac{1}{2} \sin \sigma - \dfrac{3}{2} \sin \psi . \end{cases}$$

We are interested in the unique characteristic C which at time $t = -\infty$ starts at the saddle point $(\pi, 0)$ and at time $t = \infty$ ends at the origin $(0, 0)$. Since the diagonal $u = v$ goes in the line $\sigma = \psi$ in the (σ, ψ)-plane, if we follow C from $t = -\infty$ to a time t_0 for which $\sigma = \psi$, going back to the (u, v) plane we get a geodesic starting on the axis $v = 0$ and ending on $u = v$; clearly by applying a suitable homothety we may get a geodesic ending at $u = v = \dfrac{1}{\sqrt{2}}$ and a solution to our problem. It follows that our result will be proved if we show that the characteristic C crosses the line $\sigma = \psi$ infinitely many times. This in fact is obvious, because C ends at $(0, 0)$ and it is easily checked that $(0, 0)$ is a focal singular point, or vortex, of the differential system for σ, ψ.

It may be noted that the same construction gives other examples, like for the boundary $S^2\left(\dfrac{1}{\sqrt{2}}\right) \times S^2\left(\dfrac{1}{\sqrt{2}}\right)$, with almost exactly the same result.

V. Recent progress on regularity problems

The regularity theory of minimal currents and varifolds is fundamental if we want to obtain classical solutions to variational problems. Here the theory proceeds in two main directions: one is to prove stronger and better