

AN INTEGRAL INEQUALITY IN ANALYTIC FUNCTION THEORY

Autor(en): **Haruki, Hiroshi**

Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **19 (1973)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-46294>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

AN INTEGRAL INEQUALITY IN ANALYTIC FUNCTION THEORY

by Hiroshi HARUKI

The following theorem was proved in [1]:

THEOREM A. *Suppose that $f = f(z)$ is an entire function of a complex variable z . Then the only solutions of the functional inequality*

$$(1) \quad |f((x+y)/2)| \leq (|f(x)| + |f(y)|)/2,$$

where x, y are complex variables, are $f(z) = (Az+B)^n$ and $f(z) = \exp(Az+B)$ where A, B are arbitrary complex constants and n is an arbitrary positive integer.

Now, we shall prove that (1) implies the following integral inequality:

$$(2) \quad |1/(y-x) \int_C f(z) dz| \leq (|f(x)| + |f(y)|)/2,$$

where $f = f(z)$ is an entire function of z , x, y are complex variables ($x \neq y$) and C is an arbitrary contour joining two points x and y .

To this end we shall apply the following lemma, the easy proof of which is omitted:

LEMMA 1. If $k = k(t)$ is a real-valued continuous function of a real variable t and if k is convex on $[a, b]$, then

$$1/(b-a) \int_a^b k(t) dt \leq (k(a) + k(b))/2.$$

Now, we put $g(t) = |f(x+(y-x)t)|$, where $t \in [0, 1]$ and x, y are arbitrary distinct complex constants. $g(t)$ is a real-valued continuous function on $[0, 1]$. By (1) $g(t)$ is convex on $[0, 1]$. Hence, by Lemma 1 we have

$$\int_0^1 g(t) dt \leq (g(0) + g(1))/2.$$

Therefore we have

$$\int_0^1 |f(x + (y-x)t)| dt \leq (|f(x)| + |f(y)|)/2$$

and

$$(3) \quad |1/(y-x) \int_L f(z) dz| \leq (|f(x)| + |f(y)|)/2,$$

where L is the line segment joining the two points x and y . By Cauchy's Integral Theorem (3) implies (2).

The purpose of this note is to solve (2), i.e., to prove the following

THEOREM. *Suppose that $f = f(z)$ is an entire function of z . Then the only solutions of (2) are $f(z) = (Az+B)^n$ and $f(z) = \exp(Az+B)$ where A, B are arbitrary complex constants and n is an arbitrary positive integer.*

To this end we shall apply the following two lemmas:

LEMMA 2. (See [1].) Suppose that $f = f(z), g = g(z)$ are entire functions of z . If $|f(z)| \leq |g(z)|$ holds in $|z| < +\infty$, then $f(z) = Cg(z)$ where C is a complex constant with $|C| \leq 1$.

Proof. The proof is clear from Riemann's Theorem concerning a removable singularity and Liouville's Theorem.

LEMMA 3. Suppose that $H = H(z)$ is an entire function of z . If $A(t) = |H(t \exp(i\varphi))|^2$ where t, φ are real and φ is arbitrarily fixed, then we have

$$(i) \quad A''(0) = 2\operatorname{Re}(\exp(2i\varphi) H''(0) \overline{H(0)}) + 2|H'(0)|^2,$$

$$(ii) \quad A^{(4)}(0) = 2\operatorname{Re}(\exp(4i\varphi) H^{(4)}(0) \overline{H(0)} \\ + 4\exp(2i\varphi) H^{(3)}(0) \overline{H'(0)}) + 6|H''(0)|^2.$$

Proof. Since the proof is easy, we omit it.

We may now prove our theorem.

Let $F = F(z)$ be an entire function such that $F'(z) = f(z)$. By (2) we have for all complex x, y

$$2|F(y) - F(x)| \leq |y - x|(|f(x)| + |f(y)|).$$

By a corollary of Schwarz's Inequality ($(a+b)^2 \leq 2(a^2+b^2)$, a, b real) we have

$$2|F(x) - F(y)|^2 \leq |x - y|^2(|f(x)|^2 + |f(y)|^2).$$

Replacing x, y by $x + y, x - y$, respectively, we get

$$|F(x+y) - F(x-y)|^2 \leq 2|y|^2 (|f(x+y)|^2 + |f(x-y)|^2).$$

Putting $y = t \exp(i\varphi)$ where t, φ are real we have

$$\begin{aligned} & |F(x+t \exp(i\varphi)) - F(x-t \exp(i\varphi))|^2 \\ & \leq 2t^2 (|f(x+t \exp(i\varphi))|^2 + |f(x-t \exp(i\varphi))|^2). \end{aligned}$$

Keeping x, φ arbitrarily fixed and putting

$$\begin{aligned} p(t) = & 2t^2 (|f(x+t \exp(i\varphi))|^2 + |f(x-t \exp(i\varphi))|^2) \\ & - |F(x+t \exp(i\varphi)) - F(x-t \exp(i\varphi))|^2, \end{aligned}$$

$p(t)$ is a real-valued function of t and is of course four times differentiable on $|t| < +\infty$. Further $p(t)$ is an even function of t . Hence we have

$$(4) \quad p'(0) = 0, \quad p^{(3)}(0) = 0.$$

By Lemma 3 we have

$$\begin{aligned} (5) \quad p''(0) &= 8(|f(x)|^2 - |F'(x)|^2) = 0, \\ p^{(4)}(0) &= 96 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 96 |f'(x)|^2 \\ &\quad - 32 \operatorname{Re}(\exp(2i\varphi) F^{(3)}(x) \overline{F'(x)}) \\ &= 64 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 96 |f'(x)|^2. \end{aligned}$$

$p(t)$ has a minimum at $t = 0$ ($p(t) \geq 0$ on $|t| < +\infty, p(0) = 0$). Hence, by (4), (5) we have $p^{(4)}(0) \geq 0$, or

$$2 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 3 |f'(x)|^2 \geq 0.$$

x, φ were arbitrarily fixed. An appropriate choice of φ_0 gives

$$\operatorname{Re}(\exp(2i\varphi_0) f''(x) \overline{f(x)}) = -|f''(x) \overline{f(x)}| = -|f''(x) f(x)|.$$

Hence we have in $|x| < +\infty$

$$2|f''(x) f(x)| \leq 3|f'(x)|^2,$$

and by Lemma 2

$$(6) \quad f''(x) f(x) = K f'(x)^2,$$

where K is a complex constant with $|K| \leq 3/2$.

Solving (6) and taking into account the fact that f is an entire function, we have

$$(7) \quad f(z) = (Az + B)^n \quad \text{or} \quad f(z) = \exp(Az + B),$$

where A, B are complex constants and n is an arbitrary positive integer. By Theorem A, (7) satisfies (1). Since (1) implies (2) as already proved, (7) satisfies (2). Q.E.D.

REFERENCE

- [1] HIROSHI HARUKI. On the functional inequality $|f((x+y)/2)| \leq (|f(x)| + |f(y)|)/2$, *Journal of the Mathematical Society of Japan*, 16 (1964), pp. 39-41.

(Reçu le 5 mars 1973)

Hiroshi Haruki
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario
Canada