

§ 7. Applications to divergence of Fourier series.

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satisfying $1 \leq p < 2 < q \leq \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T . Next, T is the limit in E of

$$S_r = \sum_{n=1}^r \omega_n T_{K_n}$$

as $r \rightarrow \infty$ and, since it is plain that $\text{supp } S_r \subseteq \Omega$ for every r , (ii) is easily derived. Finally, if \hat{T} were a measure μ , it would necessarily be the case that $\text{supp } \mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= |u_n * Tv_n(0)| = \left| \int_{\Gamma} \hat{u}_n \hat{v}_n d\mu \right| \\ &\leq |\mu|(\bar{\Omega}), \end{aligned}$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 REMARK. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \leq p < 2 < q \leq \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if either $q \leq 2$ or $p \geq 2$, then every $T \in L_p^q(G)$ is such that \hat{T} is a measure [and indeed a measure of the form $\psi \lambda_{\Gamma}$, where $\psi \in L_{loc}^2(\Gamma)$ if $q \leq 2$ and $\psi \in L_{loc}^p(\Gamma)$ if $p \geq 2$, and so $\psi \in L_{loc}^2(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

PART 3: APPLICATIONS TO FOURIER SERIES

§ 7. Applications to divergence of Fourier series.

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ , and λ_G the Haar measure on G , normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, \hat{f} will denote the Fourier transform of f ; for any finite subset Δ of Γ ,

$$S_{\Delta} f = \sum_{\gamma \in \Delta} \hat{f}(\gamma) \gamma \tag{7.1}$$

is the Δ -partial sum of the Fourier series of f ; and $\text{sp}(f)$ will stand for

the spectrum of f , i.e., for the support $\text{supp } \hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ of \hat{f} . The term “trigonometric polynomial” will frequently be abbreviated to “t.p.”. In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ . If Δ denotes a subset of Γ , $[\Delta]$ will stand for the subgroup of Γ generated by Δ .

By a (*convergence*) *grouping* we shall mean a sequence $\mathcal{D} = (\Delta_j)_{j \in N} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\left. \begin{aligned} \Delta_j &\subseteq \Delta_{j+1} \quad (j \in N); \\ \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 &\text{ is a subgroup of } \Gamma, \text{ said to be} \\ &\text{covered by } \mathcal{D}; \\ \text{for each } j \in N, \Delta_j &= \Omega_j + \Lambda_j, \text{ where } \Lambda_j \text{ is a} \\ &\text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ &\text{subset of } \Gamma \text{ such that } \pi|_{\Omega_j} \text{ is 1-1.} \end{aligned} \right\} \quad (7.2)$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping \mathcal{D} is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

7.2 EXAMPLES. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in \mathbb{Z}\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping \mathcal{D} covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0 . This grouping is of infinite type if and only if Γ_0 is infinite.

(ii) If G is connected, and if Γ_0 is any countable subgroup of Γ , then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R . Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in \mathbb{Z}\}$. A natural grouping \mathcal{D} covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in \mathbb{Z}, |n| \leq j\}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathcal{D} = (\Delta_j)_{j \in N}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in N}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j \rightarrow \infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f . However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\text{sp}(f) \subseteq \Gamma_0, \quad \overline{\lim_{j \rightarrow \infty} \text{Re } S_{\Delta_j} f(0)} = \infty. \quad (7.3)$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space $E = C(G)$ of continuous complex valued functions on G [with norm $\|\cdot\|$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \text{Re } S_{\Delta} f(0) = \text{Re} \int_G D_{\Delta} f d\lambda_G, \quad (7.4)$$

where D_{Δ} stands for the “Dirichlet function”

$$D_{\Delta} = \sum_{\gamma \in \Delta} \bar{\gamma}, \quad (7.5)$$

shows that the problem hinges on the existence of groupings \mathscr{D} for which

$$\rho_j = \|D_{\Delta_j}\|_1 = \int_G |D_{\Delta_j}| d\lambda_G \rightarrow \infty. \quad (7.6)$$

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ , it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j \in \mathbb{N}}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0 , we can construct a continuous f on G satisfying (7.3).

(ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ , there are groupings $\mathscr{D} = (\Delta_j)$ covering Γ_0 , in which $\Omega_j = \{0\}$ and $\Delta_j = \Delta_j$ is a finite subgroup of Γ_0 , and for which

$$f = \lim_{j \rightarrow \infty} S_{\Delta_j} f$$

uniformly on G for every continuous f satisfying $\text{sp}(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ , there is no obstacle to constructing continuous functions f such that $\text{sp}(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that $\text{sp}(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $|S_{\Delta} f(0)|$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $|S_{\Delta_j} f(0)| \rightarrow \infty$ with j .] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathcal{D} = (\Delta_j)_{j \in \mathbb{N}}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j, \nu}$ in two indeterminates over the real field (ν being a suitable fixed integer not less than 36 and p_j any positive number not less than $\|D_{\Delta_j}\|_{\infty}$) such that, for suitable unimodular complex numbers ξ_j , the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j, \nu}(D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left. \begin{aligned} \|Q_j\| &\leq 1, \text{sp}(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ S_{\Delta_j} Q_j(0) &= \int_G D_{\Delta_j} Q_j d\lambda_G \text{ is real and } \geq \frac{1}{2} \rho_j. \end{aligned} \right\} \quad (7.7)$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n \in \mathbb{N}}$ of positive integers so that

$$\left. \begin{aligned} S_{\Delta_{j_n}} Q_{j_n}(0) &\text{ is real and } > n^3, \\ j_n &< j_{n+1}, \text{sp}(Q_{j_n}) \subseteq \Gamma_0. \end{aligned} \right\} \quad (7.8)$$

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left. \begin{aligned} \text{sp}(u_n) &\subseteq \Gamma_0, \sum_{n=1}^{\infty} \|u_n\| < \infty \\ S_{\Delta_{j_n}} u_n(0) &\text{ is real and } > n. \end{aligned} \right\} \quad (7.9)$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \dots$ and specifiable sequences $(\gamma_p)_{p \in \mathbb{N}}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$\text{sp}(f) \subseteq \Gamma_0, \lim_{p \rightarrow \infty} \text{Re } S_{\Delta_{j_{n_p}}} f(0) = \infty. \quad (7.10)$$

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing $\text{Re } S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. Discussion of case (i) : G not 0-dimensional

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form

$$\Delta = \Omega + \Lambda, \quad (8.1)$$

where Ω and Λ are finite subsets of Γ such that $\pi|_{\Omega}$ is 1-1 and $\emptyset \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\|D_{\Delta}\|_1 \geq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \quad (8.2)$$

provided $N = |\Omega|$ (the cardinal number of Ω) is sufficiently large.

8.2 PROOF OF (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ/Φ . Likewise identify the dual of $K = G/H$ with Φ ([7], (24.11)).