§ 2. The construction when E is complete and first countable.

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Our final preliminary comment refers to boundedness of sets. If E is any topological linear space, a subset A of E will be said to be bounded in E if and only if to every neighbourhood U of 0 in E corresponds a number $r = r(A, U) > 0$ such that $rA = \{rx : x \in A\}$ is contained in U. If E is first countable and d is a semimetric on E defining its topology, boundedness in the above sense of a set $A \subseteq E$ must not be confused with metric boundedness [i.e., with the condition $\sup \{d(x, y) : x \in A, y \in A\} < \infty$]. It is in order to minimise the possibility of this confusion that we use the term "first countable" (an abbreviation for "satisfying the first axiom of countability") rather than "semimetrizable".

§ 2. *The construction when* E *is complete and first countable.*

In this section, where E will always denote a complete first countable (locally convex) space and P a set of bounded gauges on E, we will describe the basic construction. Let f^* denote the upper envelope of P.

If the sequence (x_n) figuring in (1.1) and (1.2) is such that $f^*(x_n) = \infty$ for some $n \in N$, no constructional problem remains. So we shall henceforth assume the contrary.

2.1 Theorem. Suppose that β and α are real numbers satisfying $\beta > \alpha > 0$ and that sequences (x_n) in E, (f_n) in P are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\lim_{n \to \infty} x_n = 0, \tag{2.2}$$

$$\sup_{n \in N} f_n(x_n) = \infty. \tag{2.3}$$

Then infinite sequences $n_1 < n_2 < \ldots$ of positive integers may be constructed such that, for every sequence (γ_n) of real numbers satisfying

$$\alpha \leqq \gamma_n \leqq \beta \quad \text{for every } n \in N, \tag{2.4}$$

the series

$$\sum_{\nu \in N} \gamma_\nu x_{n_\nu} \tag{2.5}$$

is normally convergent in E, and

$$f^*(x) \geqq \lim_{\nu \to \infty} f_{n_\nu}(x) = \infty \tag{2.6}$$

for each sum x of (2.5).

2.2 CONSTRUCTION AND PROOF. Let (σ_ν) be an increasing sequence of continuous seminorms on E which define its topology. By initial passage to suitable subsequences, we may and will assume that (2.2) and (2.3) hold in the stronger form:

$$\sum_{n\in N} \sigma_n(x_n) < \infty, \tag{2.2'}$$

$$\lim_{n\to\infty} f_n(x_n) = \infty. \tag{2.3'}$$

[To do this, define $n_\nu \in N$ for $\nu \in N$ by induction in such a way that $n_1 < n_2 < \ldots$,

$$\sigma_\nu(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{and} \quad f_{n_\nu}(x_{n_\nu}) > \nu \tag{2.7}$$

for all $\nu \in N$. This is possible since by (2.2) we can determine $\mathring{n}_1 \in N$ such that $\sigma_1(x_n) \leqq 2^{-1}$ if $n \geqq \mathring{n}_1$, and then, by (2.3) and the fact that each $f \in P$ is finite valued, there exists $n \geqq \mathring{n}_1$ such that $f_n(x_n) > 1$; denote the smallest such $n \geqq \mathring{n}_1$ by n_1. When $n_1 < n_2 < \ldots n_j$ have been determined so that (2.7) holds for $1 \leqq \nu \leqq j$, find (see (2.2)) an integer $\mathring{n}_{j+1} > n_j$ such that $\sigma_{j+1}(x_n) \leqq 2^{-j-1}$ if $n \geqq \mathring{n}_{j+1}$. Then (2.3) shows that there exists an integer $n \geqq \mathring{n}_{j+1}$ such that $f_n(x_n) > j+1$; put n_{j+1} for the smallest such integer $n \geqq \mathring{n}_{j+1}$.]

So now we assume (2.1), (2.2′) and (2.3′) and define one sequence $n_1 < n_2 < \ldots$ of the required type in the following manner. (Other possibilities are discussed in Remark 2.3 (2) below.) Let n_1 be the smallest $n \in N$ such that

$$f_n(x_n) \geqq \beta\alpha^{-1};$$

n_1 may be determined by (2.3′). Suppose that ν is a positive integer and that positive integers $n_1 < n_2 < \ldots < n_\nu$ have been defined so that

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu,$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \sum_{1\leqq j<\nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu.$$

[An empty sum is defined to be 0; then the conditions are all satisfied when $\nu = 1$.] Then (2.2′), (2.3′) and the fact that each $f \in P$ is finite-valued imply that there exists an integer $n > n_\nu$ which satisfies

$$f_{n_j}(x_n) \leqq 2^{-\nu-1} \quad \text{whenever} \quad 1 \leqq j < \nu+1,$$

$$f_n(x_n) \geqq \beta\alpha^{-1} \sum_{1\leqq j<\nu+1} f_n(x_{n_j}) + \beta\alpha^{-1}(\nu+1);$$

let $n_{\nu+1}$ be the smallest such n. We then have for each $\nu \in N$:

$$n_\nu < n_{\nu+1},$$

$$f_{n_j}(x_{n_\nu}) \leqq 2^{-\nu} \quad \text{whenever} \quad 1 \leqq j < \nu, \tag{2.8}$$

$$f_{n_\nu}(x_{n_\nu}) \geqq \beta\alpha^{-1} \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\alpha^{-1}\nu. \tag{2.9}$$

By (2.2′) and (2.4), the sum (2.5) is normally convergent in E. Let x be any sum of this series. To establish (2.6), write

$$x = u_\nu + \gamma_\nu x_{n_\nu} + v_\nu,$$

where $u_\nu = \sum_{1 \leqq j < \nu} \gamma_j x_{n_j}$ and v_ν is a sum of the series $\Sigma_{j>\nu} \gamma_j x_{n_j}$. Thus $\gamma_\nu x_{n_\nu} = x - u_\nu - v_\nu$, and so

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(\gamma_\nu x_{n_\nu}) \leqq f_{n_\nu}(x) + f_{n_\nu}(u_\nu) + f_{n_\nu}(v_\nu). \tag{2.10}$$

Now, by (2.4),

$$f_{n_\nu}(u_\nu) \leqq \beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}); \tag{2.11}$$

and, by (2.4), (2.8) and the fact that each f_n is bounded, hence continuous,

$$f_{n_\nu}(v_\nu) \leqq \beta \textstyle\sum_{j>\nu} f_{n_\nu}(x_{n_j}) \leqq \beta \sum_{j>\nu} 2^{-j} = \beta 2^{-\nu}. \tag{2.12}$$

By (2.10), (2.11) and (2.12)

$$\alpha f_{n_\nu}(x_{n_\nu}) \leqq f_{n_\nu}(x) + \beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu},$$

and so, by (2.9),

$$\beta \textstyle\sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta\nu \leqq f_{n_\nu}(x) + \beta \sum_{1 \leqq j < \nu} f_{n_\nu}(x_{n_j}) + \beta 2^{-\nu}.$$

Hence

$$f_{n_\nu}(x) \geqq \beta(\nu - 2^{-\nu}),$$

which proves (2.6) and the construction is complete.

2.3 Remarks. (1) If it is known that

$$D = \{x \in E : f^*(x) < \infty\}$$

is dense in E, and if (x_n) and (f_n) satisfy (2.2) and (2.3), we can approximate each x_n so closely by an element y_n of D that (2.2) and (2.3) are left intact on replacing x_n by y_n. The hypotheses (2.1)—(2.3) are satisfied when x_n is everywhere replaced by y_n.

(2) If it be supposed that (2.2′) holds and that sequences (A_n), $(B_{n,r})$ and (C_n) are known such that $\lim_{n\to\infty} B_{n,r} = 0$ for every $r \in N$, $\lim_{n\to\infty} C_n = \infty$,

$$f^*(x_1) + \dots + f^*(x_n) \leqq A_n,$$

$$\max_{1\leqq j\leqq r} f_j(x_n) \leqq B_{n,r},$$

$$f_n(x_n) \geqq C_n,$$

then it is easy to specify a function $\phi_{\alpha,\beta} : N \times N \to N$ in terms of (A_n), $(B_{n,r})$ and (C_n) such that (2.4) and (2.5) yield (2.6) for every sequence (n_ν) such that $C_{n_1} \geqq \beta\alpha^{-1}$ and $n_{\nu+1} \geqq \phi_{\alpha,\beta}(n_\nu, \nu)$ for every $\nu \in N$.

(3) Local convexity of E is not essential in 2.1 and 2.2. In the contrary case one may proceed by introducing an invariant semimetric $(x, y) \mapsto |x-y|$ defining the topology of E, much as in [2], proof of Theorem 6.1.1, or [15], Chapitre I, § 3, No. 1. Normal summability in E of a series $\sum_{n\in N} z_n$ of elements of E may then be taken to mean the convergence of $\sum_{n\in N} |z_n|$. In place of (2.2′) arrange that

$$\sum_{n\in N} |\beta x_n| < \infty,$$

which will ensure the normal convergence in E of (2.5) whenever (2.4) holds (E being assumed to be complete). The rest of the proof and construction proceeds as before.

This method could, of course, be used when E is locally convex (and first countable and complete); we have not done so because the seminorms σ_n are usually more manageable in practice.

(4) A useful variant of 2.1 may be stated in the following terms.

2.4 Suppose given real numbers $\beta > \alpha > 0$ and sequences (x_n) in E and (f_n) in P such that

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\{x_n : n \in N\} \quad \text{is bounded in } E, \tag{2.2″}$$

$$\sup_{n\in N} f_n(x_n) = \infty. \tag{2.3}$$

Then one can construct a sequence (λ_n) of real numbers with the following properties:

$$\lambda_n \geqq 0, \sum_{n\in N} \lambda_n < \infty; \tag{2.13}$$

for every sequence (γ_n) satisfying (2.4) the series

$$\sum_{n \in N} \gamma_n \lambda_n x_n \tag{2.14}$$

is normally convergent in E; and

$$f^*(x) = \infty \tag{2.15}$$

for every sum x of the series (2.14).

In the sequel we shall denote by $l^1_+(N)$ the set of sequences (λ_n) satisfying (2.13).

Proof. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2}\, x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_ν) of positive integers such that (2.4) entails that the series

$$\sum_{\nu \in N} \gamma_\nu\, n_\nu^{-2}\, x_{k_{n_\nu}} \tag{2.16}$$

is normally convergent in E and that (2.15) holds for every sum x of (2.16). It thus suffices to define λ_n to be n_ν^{-2} when $n = k_{n_\nu}$ for some $\nu \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. *The construction when* E *is sequentially complete*

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E, and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1), (2.2″) and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

Proof. Consider the continuous linear map T of $l^1(N)$ into E defined by

$$T\xi = \sum_{n \in N} \xi_n\, x_n.$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n, and f_n by $f_n \circ T$.

The following corollary will find application in §§ 5 and 6 below.