§ 1. Introduction and preliminaries

A NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

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General Introduction

This paper is partly pedagogical and expository. Thus Part 1 (§§ 1-4) presents a naively constructive approach to boundedness principles. Although this construction leads to results differing but slightly from the standard versions, we feel that this approach (which can be followed with no overt reference to category, barrelled spaces, and so on) offers some pedagogical and expository advantages. We emphasise that the level of constructivity is naive and not fundamental.

The remainder of the paper consists of applications of the constructive procedure. In Part 2 (§§ 5, 6) the applications yield improvements of recent results due to Price and to Gaudry concerning multipliers. In Part 3 (§§ 7-10) the applications are to convergence and divergence of Fourier series of continuous functions on compact Abelian groups. These results (which may be known to the afficionados but which, as far as we know, have not been published hitherto) characterise those compact Abelian groups having the property that every continuous function has a convergent Fourier series; and, in the remaining cases, applies the general method of Part 1 to construct continuous functions with divergent Fourier series.

Part 1: Boundedness principles

§ 1. *Introduction and preliminaries*

Let E denote a locally convex space and P a set of bounded gauges on E; that is, each $f \in P$ is a function with domain E and range a subset of $[0, \infty)$ such that

$$f(x+y) \leqq f(x) + f(y) \quad (x, y \in E),$$

$$f(\alpha x) = \alpha f(x) \quad (x \in E, \alpha > 0),$$

(so that $f(0) = 0$) and f is bounded on every bounded subset of E. In all cases, if f is continuous, then it is bounded; the converse is true if E is bornological ([2], p. 477). Note also that any seminorm is a positive gauge function; so too are $Re^+ u = \sup (Re\, u, 0)$ and $Im^+ u = \sup (Im\, u, 0)$, whenever u is a real-linear functional on E.

The boundedness principles discussed in this paper are those which assert that, granted suitable conditions on E, if the upper envelope f^* of P is finite valued, then f^* (which is evidently a gauge) is also bounded (cf. [2], Ch. 7).

It is customary to prove this type of boundedness principle (with continuous seminorms in place of bounded gauges) by appeal to assumed properties of E (for example, that it be second category, or barrelled, or sequentially complete and infrabarrelled) of a sort which renders the proof almost effortless.

One indirect use of boundedness principles aims at establishing the existence of misbehaviour, leaving aside any attempt to locate any specific instance thereof (cf. Banach's famous "principe de condensation des singularités"). We are here referring to situations in which a sequence (x_n) in E is known which satisfies

$$(x_n) \text{ is bounded (or convergent-to-zero) in } E \tag{1.1}$$

and

$$\sup_{n \in N} f^*(x_n) = \infty, \tag{1.2}$$

and an appeal to a boundedness principle is then made to infer the existence of one or more elements x of E satisfying

$$f^*(x) = \infty. \tag{1.3}$$

[The argument is simply that the negation of (1.3) implies, via a boundedness principle, that f^* is bounded (or continuous), and that this involves a contradiction of the conjunction of (1.1) and (1.2).]

The alternative to be advocated in this paper amounts to seeking a constructive procedure (involving no appeal to boundedness principles) leading from (1.1) and (1.2) to specified elements x satisfying (1.3). To do this seems all the more natural when, as is often the case, a fair amount of effort has already been expended in constructing a sequence (x_n) satisfying (1.1) and (1.2). Moreover, granted such a procedure, general boundedness principles can be derived quite easily (see §§ 3 and 4). This incidental approach to boundedness principles appears to be at least as successful as the customary one.

A construction of the desired type (a special case of which was subsequently located in the Appendix to [6]; see also [12], Solution 20 in [13], and [16]) is easily describable if E is complete and first countable (see § 2 below). The procedure is then extendible to sequentially complete spaces E (see § 3), and from this follows at once the corresponding version of the boundedness principle applying to bounded gauges (see § 4). Continuity of f^* follows under appropriate additional conditions.

Since we shall be working with gauge functions which are assumed to be merely bounded (rather than continuous), the usual standard passage from a non-Hausdorff space to its Hausdorff quotient is not generally available. For this reason, it seems worthwhile to formulate the results without assuming that E is Hausdorff. (If E is bornological—for example, first countable ([2], 6.1.1 and 7.3.2)—there is no problem.)

We shall write N for $\{1, 2, ...\}$; and the sequence $(u_n)_{n \in N}$ will often be written briefly as (u_n).

If E is any locally convex space and (x_n) a sequence of elements of E, the series $\sum_{n \in N} x_n$ or $\sum_{n=1}^{\infty} x_n$ is said to be *normally summable in* E if $\sum_{n \in N} \sigma(x_n) < \infty$ for every continuous seminorm σ on E. The series $\sum_{n \in N} x_n$ is said to be *convergent in* E and to have $x \in E$ as a sum, written $x \sim \sum_{n \in N} x_n$, if

$$\lim_{k \to \infty} \sigma(x - \textstyle\sum_{n=1}^{k} x_n) = 0$$

for every continuous seminorm σ on E; the set of sums of a given convergent series form precisely one equivalence class modulo $\{0\}^-$. A series which is both normally summable and convergent in E is said to be *normally convergent in E*, or to *converge normally in E*. If E is sequentially complete, any series which is normally summable in E is normally convergent in E.

Two comments regarding the hypotheses imposed upon E are worth making at the outset. In the first place, we have concentrated on the locally convex case, with only Remarks 2.3 (3), 3.3 (3) and 4.2 (2) referring to the alternative, the reason being that this is by far the most important case for applications. Accordingly, throughout §§ 2-4, *E will* (except where the contrary is explicitly indicated) *be assumed to be locally convex.*

In the second place, it would suffice for subsequent developments to have Theorem 2.1 established for Banach spaces (and even merely for the familiar Banach space $l^1(N)$). However, only limited economy is gained by dealing with this special case alone and it seems best to retain a degree of generality which allows a more direct and explicit approach in the case of (say) Fréchet spaces.

Our final preliminary comment refers to boundedness of sets. If E is any topological linear space, a subset A of E will be said to be bounded in E if and only if to every neighbourhood U of 0 in E corresponds a number $r = r(A, U) > 0$ such that $rA = \{rx : x \in A\}$ is contained in U. If E is first countable and d is a semimetric on E defining its topology, boundedness in the above sense of a set $A \subseteq E$ must not be confused with metric boundedness [i.e., with the condition $\sup \{d(x, y) : x \in A, y \in A\} < \infty$]. It is in order to minimise the possibility of this confusion that we use the term "first countable" (an abbreviation for "satisfying the first axiom of countability") rather than "semimetrizable".

§ 2. *The construction when* E *is complete and first countable.*

In this section, where E will always denote a complete first countable (locally convex) space and P a set of bounded gauges on E, we will describe the basic construction. Let f^* denote the upper envelope of P.

If the sequence (x_n) figuring in (1.1) and (1.2) is such that $f^*(x_n) = \infty$ for some $n \in N$, no constructional problem remains. So we shall henceforth assume the contrary.

2.1 THEOREM. Suppose that β and α are real numbers satisfying $\beta > \alpha > 0$ and that sequences (x_n) in E, (f_n) in P are such that:

$$f^*(x_n) < \infty \quad \text{for every } n \in N, \tag{2.1}$$

$$\lim_{n \to \infty} x_n = 0, \tag{2.2}$$

$$\sup_{n \in N} f_n(x_n) = \infty. \tag{2.3}$$

Then infinite sequences $n_1 < n_2 < \ldots$ of positive integers may be constructed such that, for every sequence (γ_n) of real numbers satisfying

$$\alpha \leqq \gamma_n \leqq \beta \quad \text{for every } n \in N, \tag{2.4}$$

the series

$$\sum_{\nu \in N} \gamma_\nu x_{n_\nu} \tag{2.5}$$

is normally convergent in E, and

$$f^*(x) \geqq \lim_{\nu \to \infty} f_{n_\nu}(x) = \infty \tag{2.6}$$

for each sum x of (2.5).