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A sixteen-relator presentation of an infinite hyperbolic Kazhdan group

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Abstract. We provide an explicit presentation of an infinite hyperbolic Kazhdan group with 4 generators and 16 relators of length at most 73. That group acts properly and cocompactly on a hyperbolic triangle building of type $(3, 4, 4)$. We also point out a variation of the construction that yields examples of lattices in $\tilde{A}_2$-buildings admitting non-Desarguesian residues of arbitrary prime power order.

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1. Hyperbolic Kazhdan groups

The existence of infinite Gromov hyperbolic groups enjoying Kazhdan's property (T) has been known since the origin of the theory of hyperbolic groups, as a combination of the following results.

- Every simple Lie group possesses a cocompact lattice, by [Bor];
- the rank one simple Lie groups $\mathrm{Sp}(n, 1)$ (with $n \geq 2$) and F_4^{-20} have (T), by [Kos] (see also [BdlHV, §3.3]);
- if a locally compact group G has Property (T), then so does every lattice Γ in G by [Kaz] (see also [BdlHV, Theorem 1.7.1]);
- a cocompact lattice in a rank one simple Lie group is Gromov hyperbolic, since it is virtually the fundamental group of a closed Riemannian manifold of negative sectional curvature, see [Gro].

Those results imply in particular the existence of a negatively curved closed manifold M of dimension 8 whose fundamental group $\pi_1(M)$ has Kazhdan's property (T) (namely M is covered by the symmetric space of $\mathrm{Sp}(2, 1)$). I am not aware of any known explicit presentation of the fundamental group $\pi_1(M)$

in that case. This is a very interesting and natural problem. It is also natural to ask whether the fundamental group of a negatively curved closed manifold M of dimension < 8 can have property (T). If M has dimension 2 or 3, then the Hyperbolization Theorem (see [AFW, Theorem 1.7.5] and references therein) ensures that the fundamental group $\pi_1(M)$ is a lattice in $O(2,1)$ or $O(3,1)$. Therefore it cannot be a Kazhdan group by [BdlHV, Theorem 2.7.2] (see also [Fuj] for a more general result on the failure of Property (T) for 3-manifold groups). It is currently unknown whether a negatively curved closed manifold of dimension $4, 5, 6$ or 7 can have a Kazhdan fundamental group. Misha Kapovich pointed out to me that the related problem of finding objects of either of the following kinds, is also open:

- a nonpositively curved closed manifold, not homeomorphic to a locally symmetric space, and with a Kazhdan fundamental group;
- a Kazhdan Poincaré duality group not isomorphic to a lattice in a connected Lie group.

The possibility to write down an explicit presentation of an infinite hyperbolic Kazhdan group was first realized in [BS, Corollary 2], where the geometric approach to Property (T) via the spectral gap of finite graphs is exploited (see [BdlHV, Chapter 5] for an exposition of that approach including a historical account). The graphs used in [BS] are certain Cayley graphs of $\mathrm{SL}_2(\mathbf{Z}/n\mathbf{Z})$, which satisfy the required spectral gap condition for n sufficiently large. An alternative source of finite Cayley graphs that enjoy the required spectral condition is suggested by Alain Valette in his review of [BS], but I am not aware of any reference where that suggestion was incarnated into an explicit presentation of a hyperbolic Kazhdan group. A different construction is highlighted by Marc Bourdon in [Bou, §1.5.3]. It gives rise to cocompact lattices in certain Gromov hyperbolic triangle buildings, and also relies on the geometric approach to Property (T). The advantage is that the finite graphs on which the spectral gap condition is tested are finite generalized polygons, and the eigenvalues of their incidence matrix is explicitly known by classical results from [FH]. Nevertheless, the corresponding group presentations one obtains from that construction take several hundreds relations. The variations on Bourdon's construction described in [Świ] also seem to require a rather large number of relators. Other examples of infinite hyperbolic Kazhdan groups are studied in [LMW], but no explicit short presentation is recorded there.

Cornelia Druţu asked me whether it was possible to use buildings in order to construct an explicit short presentation of an infinite hyperbolic group with Kazhdan's Property (T). As explained in [DK, Section 19.8]: "*while 'generic' finitely presented groups are infinite and satisfy Property (T), finding explicit and* *reasonably short presentations presents a bit of a challenge*". In that context, targeting hyperbolic buildings is especially natural in view of the fact that there exist 5-relator presentations of infinite Kazhdan groups acting properly and cocompactly on buildings of type $\tilde{A}_2$, see [Ess, Examples following Theorem 5.8]. Note that those groups cannot be hyperbolic since they are quasi-isometric to a 2-dimensional Euclidean building. The shortest presentation I could find in attempting to answer Cornelia Druţu's question is the following.

Theorem 1. *The group*

$$\begin{aligned} E = \langle x, y, z, t, r \mid\ & x^7, y^7, [x, y]z^{-1}, [x, z], [y, z], \\ & t^2, r^{73}, trtr, \\ & [x^2yz^{-1}, t], [xyz^3, tr], [x^3yz^2, tr^{17}], \\ & [x, tr^{-34}], [y, tr^{-32}], [z, tr^{-29}], \\ & [x^{-2}yz, tr^{-25}], [x^{-1}yz^{-3}, tr^{-19}], [x^{-3}yz^{-2}, tr^{-11}]\rangle \end{aligned}$$

is an infinite Gromov hyperbolic group enjoying Kazhdan's Property (T). It is virtually torsion-free, and acts faithfully, properly, cocompactly (not type-preservingly) on a thick hyperbolic triangle building of type $(3, 4, 4)$. *In particular* E *is quasi-isometrically rigid by* [Xie].

In view of the relation $[x, y] = z$, the generator z is redundant, and the presentation of E given in Theorem 1 is equivalent to a presentation with 4 generators and 16 relators. This modification increases the length of some of the relators, but one checks that the maximal length of a relator in that 16-relator presentation of E remains equal to 73.

To prove that E is infinite and hyperbolic, we identify E as the fundamental group of a simple complex $E(\mathcal{Y})$ of finite groups, in the sense of [BH, Chapter II.12]. The complex in question is described in Section 2. In verifying that this complex is developable, we provide a complete description of the link of every vertex. All of them happen to be *incidence graphs of generalized polygons*, i.e., bipartite graphs whose diameter is equal to half of the girth. Since the spectrum of the Laplace operator on such graphs is known by the work of Feit–Higman [FH], we may invoke a criterion due to Izhar Oppenheim [Opp] in order to establish that E has Property (T). The proof of Theorem 1 is completed in Section 7.

A local characterization of buildings due to Jacques Tits [Tit] ensures that the global development of the complex $E(\mathcal{Y})$, which is a simply connected 2-dimensional simplicial complex that we denote by $D(\mathcal{Y})$, is a non-thick hyperbolic triangle building of type $(2, 4, 6)$. A canonical procedure, described in Section 7 and consisting in discarding the edges contained in exactly two 2-simplices, allows us to view $D(\mathcal{Y})$ as a thick hyperbolic triangle building of type $(3,4,4)$. This enables us to invoke a result of Xiangdong Xie [Xie] ensuring that E is quasi-isometrically rigid.

The final section of the paper records several variations of the construction giving numerous additional examples of infinite hyperbolic Kazhdan groups, as well as groups acting properly cocompactly on $\tilde{A}_2$-buildings with non-Desarguesian residue planes. It also contains an infinite hyperbolic group, denoted by E_3, admitting a presentation much shorter than the presentation of E from Theorem 1; the question whether E_3 satisfies Property (T) is open (see Question 13).

The paper has been written in such a way that it should be accessible to a reader without any prior knowledge of the theory of buildings. Buildings appear in Section 6, whose purpose is to clarify the connection between this paper and the work of Jan Essert [Ess]. However, the proof of Theorem 1 does not rely on that section. The only prerequisite needed for the proof of Theorem 1 is some familiarity with the theory of non-positively curved simple complexes of finite groups, developed in [BH, Chapter II.12].

2. A simple complex of finite groups

We assume that the reader is familiar with the terminology and notation from [BH, Chapter II.12]. Let Δ be a geodesic triangle with angles $\pi/6, \pi/4, \pi/2$ in the real hyperbolic plane. Let $\mathcal{Y}$ be the 2-dimensional simplicial complex on 11 vertices, denoted by $a, b, c_1, \dots, c_9$, obtained by glueing 9 isometric copies of Δ along their hypothenuse $[a,b]$, as depicted in Figure 1. Hence $\mathcal{Y}$ is a piecewise hyperbolic complex. In each triangular face abc_i, the angle at a is $\pi/6$ and the angle at b is $\pi/4$.

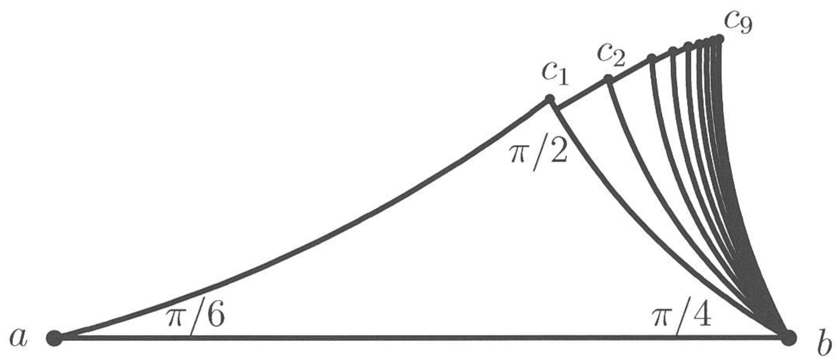

FIGURE 1
The complex $\mathcal{Y}$

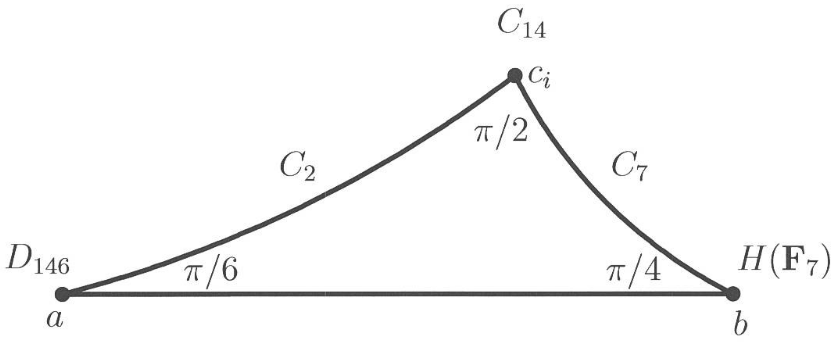

FIGURE 2
The vertex and edge groups in $E(\mathcal{Y})$

Next we construct a simple complex of groups $E(\mathcal{Y})$. First we define the local groups as follows (see Figure 2).

- The vertex group $E_a = \langle t, r \mid t^2, r^{73}, trtr \rangle$ is the dihedral group of order 146.

- The vertex group $E_b = \langle x, y, z \mid x^7, y^7, [x, y]z^{-1}, [x, z], [y, z] \rangle$ is the Heisenberg group over $\mathbf{F}_7$, of order 343. Notice that the relation $z^7 = 1$ follows from the others in the group E_b.

- The vertex group $E_{c_i} = \langle w_i \mid w_i^{14} \rangle$ is the cyclic group of order 14 for $i = 1, \ldots, 9$.

- The edge groups E_{ac_i} (resp. E_{bc_i}) are cyclic of order 2 (resp. 7).

- The edge group E_{ab} and the face groups E_{abc_i} are trivial.

The glueing homomorphisms are defined as follows.

- For $i = 1, \ldots, 9$, we identify E_{ac_i} with $\langle w_i^7 \rangle \leq E_{c_i}$ and E_{bc_i} with $\langle w_i^2 \rangle \leq E_{c_i}$.

- For $i = 1, \ldots, 9$, the homomorphism $E_{ac_i} \to E_a$ maps w_i^7 to t, tr, tr^{17}, $tr^{-34}, tr^{-32}, tr^{-29}$, tr^{-25}, tr^{-19} and tr^{-11} respectively.

- For $i = 1, \ldots, 9$, the homomorphism $E_{bc_i} \to E_b$ maps w_i^2 to x^2yz^{-1}, xyz^3, x^3yz^2, x, y, z, $x^{-2}yz, x^{-1}yz^{-3}$ and $x^{-3}yz^{-2}$ respectively.

It follows from [BH, §II.12.12] that the fundamental group $\widehat{E(\mathcal{Y})}$ admits the following presentation:

$$\begin{aligned}\widehat{E(\mathcal{Y})} = \langle x, y, z, t, r, w_1, \ldots, w_9 \mid\ & x^7, y^7, [x,y]z^{-1}, [x,z], [y,z],\\ & t^2, r^{73}, trtr,\\ & w_1^{14}, w_2^{14}, \ldots, w_9^{14},\\ & \begin{array}{rclrclrcl} t &=& w_1^7, & tr &=& w_2^7, & tr^{17} &=& w_3^7,\\ tr^{-34} &=& w_4^7, & tr^{-32} &=& w_5^7, & tr^{-29} &=& w_6^7,\\ tr^{-25} &=& w_7^7, & tr^{-19} &=& w_8^7, & tr^{-11} &=& w_9^7,\\ x^2yz^{-1} &=& w_1^2, & xyz^3 &=& w_2^2, & x^3yz^2 &=& w_3^2,\\ x &=& w_4^2, & y &=& w_5^2, & z &=& w_6^2,\\ x^{-2}yz &=& w_7^2, & x^{-1}yz^{-3} &=& w_8^2, & x^{-3}yz^{-2} &=& w_9^2 \rangle. \end{array}\end{aligned}$$

It is straightforward to check that the group E from Theorem 1 is isomorphic to $\widehat{E(\mathcal{Y})}$ by observing the existence of natural homomorphisms $\widehat{E(\mathcal{Y})} \to E$ and $E \to \widehat{E(\mathcal{Y})}$ that are inverse of one another.

In order to prove that E is infinite hyperbolic, we will rely on Theorem II.12.28 from [BH], which will ensure that the complex $E(\mathcal{Y})$ is developable. To verify the hypotheses of that result, we need to understand the shape of the local developments of that complex at every vertex (see [BH, §II.12.24] for the definition of the local development). To that end, the following terminology will be useful.

Given a group G and a collection $\{P_i \mid i \in I\}$ of subgroups of G, the *bipartite coset graph* of G with respect to $\{P_i \mid i \in I\}$ is the bipartite graph whose vertex set is the disjoint union of G with $\bigsqcup_{i \in I} G/P_i$, and where the element $g \in G$ forms an edge with the coset hP_i if and only if $g \in hP_i$. That graph is connected if and only if G is generated by the set $\bigcup_{i \in I} P_i$.

The following observation follows directly from the definitions.

Lemma 2. *In the complex $E(\mathcal{Y})$, the link at a in the local development around a is isomorphic, as a simplicial graph, to the bipartite coset graph of the finite group $E_a = \langle t, r \mid t^2, r^{73}, trtr\rangle$ with respect to the 9 cyclic subgroups generated by generated by t, tr, tr^{17}, tr^{-34}, tr^{-32}, tr^{-29}, tr^{-25}, tr^{-19} and tr^{-11}.*

Similarly, the link at b in the local development around b is isomorphic, as a simplicial graph, to the bipartite coset graph of $E_b = \langle x, y, z \mid x^7, y^7, [x,y]z^{-1}, [x,z], [y,z]\rangle$ with respect to the 9 cyclic subgroups generated by x^2yz^{-1}, xyz^3, x^3yz^2, x, y, z, $x^{-2}yz$, $x^{-1}yz^{-3}$ and $x^{-3}yz^{-2}$. □

We now investigate those graphs in more detail.

3. Projective planes and dihedral groups

We review some basic notions from the theory of projective planes. For a comprehensive account, we refer to the book [HP].

A *generalized triangle* is a bipartite graph with diameter 3 and girth 6. The incidence graph of every projective plane of order q is a generalized triangle all of whose vertices have degree $q+1$; conversely, every generalized triangle whose vertex degrees are all $q+1$, with $q \geq 2$, is the incidence graph of a unique projective plane of order q.

A *difference set* in a group G is a subset $\mathcal{D}$ of G such that every non-trivial element g of G can be written in a unique way as $g = \sigma^{-1}\tau$ with $\sigma, \tau \in \mathcal{D}$. Notice that G must have order q^2+q+1 where $q = |\mathcal{D}|-1$. The following special instance is directly related to the group E:

Example 3. The set

$$\begin{aligned}\mathcal{D} &= \{0, 1, 17, 39, 41, 44, 48, 54, 62\} \\ &= \{0, 1, 17, -34, -32, -29, -25, -19, -11\}\end{aligned}$$

is a difference set in the cyclic group $\mathbf{Z}/73\mathbf{Z}$.

Difference sets in groups are tightly connected with projective planes. Details may be consulted in [HP, § XIII.5] or in [Dem, pp. 105–106]. For our purposes, we need the following result, showing that a difference set in a cyclic group of order $n = q^2+q+1$ allows us to associate a projective plane of order q to the dihedral group of order $2n$. Such a connection was first highlighted by Ivanov–Iofinova in [II, Lemma 3.2].

Proposition 4. *Let $q \geq 2$ be an integer and let $n = q^2+q+1$. Let $D_{2n} = \langle r, t \mid r^n, t^2, trtr\rangle$ be the dihedral group of order $2n$, and let $\mathcal{D}$ be a difference set in the cyclic group $\mathbf{Z}/n\mathbf{Z}$. Then:*

(i) *The Cayley graph of D_{2n} with respect to the set $\{tr^\sigma \mid \sigma \in \mathcal{D}\}$ is the incidence graph of a projective plane of order q.*

(ii) *The bipartite coset graph of D_{2n} with respect to the subgroups $\{\langle tr^\sigma\rangle \mid \sigma \in \mathcal{D}\}$ is the first barycentric subdivision of the incidence graph of a projective plane of order q.*

Proof. Since any reflection in D_{2n} has non-trivial image in the quotient $D_{2n}/\langle r\rangle$, it follows that any loop in the Cayley graph $\mathcal{G}$ of D_{2n} with respect to the set $\{tr^\sigma \mid \sigma \in \mathcal{D}\}$ has even length. In particular $\mathcal{G}$ is bipartite. If $\mathcal{G}$ contains a loop of length 4 through the identity, then there exist $\sigma_1, \dots, \sigma_4 \in \mathcal{D}$ with $1 = tr^{\sigma_1}tr^{\sigma_2}tr^{\sigma_3}tr^{\sigma_4}$. Hence $r^{-\sigma_1+\sigma_2}r^{-\sigma_3+\sigma_4} = 1$. Since $\mathcal{D}$ is a difference set, we must have $\sigma_1 = \sigma_4$ and $\sigma_2 = \sigma_3$, so that the loop was a backtracking path. Thus $\mathcal{G}$ has girth at least 6. Observing that $\mathcal{G}$ is a vertex-transitive bipartite graph of degree $q+1$, we infer that the total number of vertices at distance exactly 2 from the identity vertex in $\mathcal{G}$ is $q(q+1)$. Since the total number of vertices of $\mathcal{G}$ is $2(q^2+q+1)$ and since $\mathcal{G}$ is bipartite, we deduce that $\mathcal{G}$ has diameter 3 and girth 6. This proves assertion (i). Assertion (ii) follows from (i) since the bipartite coset graph in question is the first barycentric subdivision of $\mathcal{G}$. □

Corollary 5. *The link at the vertex a in the local development of the complex $E(\mathcal{Y})$ around a is isomorphic, as a simplicial graph, to the first barycentric subdivision of a generalized triangle of order* 8. *In particular it has diameter* 6 *and girth* 12.

Proof. This follows directly from Lemma 2 and Proposition 4. □

Remark 6. It is a famous open problem to determine the integers $n > 3$ such that the cyclic group of order n contains a difference set. Clearly n must be of the form $n = q^2 + q + 1$ for some integer $q \geq 2$. A sufficient condition is that q be a prime power: see the Corollary to Theorem 2.64, together with Lemma 13.12, in [HP]. The Prime Power Conjecture predicts that this sufficient condition is also necessary.

4. Generalized quadrangles and Heisenberg groups

We recall that a graph is the incidence graph of a *generalized quadrangle* if and only if it is bipartite, has diameter 4 and girth 8. The *order* of a finite generalized quadrangle is the pair (s,t) such that the vertex degrees of the incidence graph of the quadrangle are $s+1$ and $t+1$.

The following observation is closely related to a result of W. Kantor [Kan1, Theorem 2]. It allows one to recognize when a bipartite coset graph (which is indeed a bipartite graph) is the incidence graph of a generalized quadrangle.

Proposition 7. *Let $\mathcal{G}$ be the bipartite coset graph of a group G with respect to a collection $\{P_i \mid i \in I\}$ of subgroups. Assume that $|I| \geq 2$ and that $P_i \neq \{e\}$ for all $i \in I$.*

(i) *If $P_i \cap P_j = \{1\}$ for all distinct $i, j \in I$, then $\mathcal{G}$ has girth ≥ 6.*

(ii) *If $P_i P_j \cap P_k = \{1\}$ for all distinct $i, j, k \in I$, then $\mathcal{G}$ has girth ≥ 8.*

(iii) *Let $t = |I| - 1$ and suppose that $s = |P_i| - 1$ for all $i \in I$. If the condition (ii) holds and if in addition G is finite of order $|G| = (1+s)(1+st)$, then $\mathcal{G}$ is the incidence graph of a generalized quadrangle of order (s,t).*

Proof. The proof is a direct computation similar to the proof of Proposition 4. □

The following consequence allows one to recover a family of finite generalized quadrangles that is well-known to the experts; it was first discovered by S. Payne [Pay]. The right choice of $p+2$ cyclic subgroups was recorded in [Ess, Theorem 3.10].

Corollary 8. *Let p be an odd prime and $H(\mathbf{F}_p) = \langle x, y, z \mid x^p, y^p, [x, y]z^{-1}, [x, z], [y, z]\rangle$ be the Heisenberg group over $\mathbf{F}_p$. Then the bipartite coset graph of $H(\mathbf{F}_p)$ with respect to the collection $\{\langle x\rangle, \langle z\rangle\} \cup \{\langle x^a yz^{-\frac{a}{2}}\rangle \mid a = 0, \dots p-1\}$ of $p+2$ cyclic subgroups of order p is the incidence graph of a generalized quadrangle of order $(p-1, p+1)$.*

Proof. For all integers $a, b, n \in \mathbf{Z}$, we have

$$(x^a y^b)^n = x^{na} y^{nb} z^{-\frac{(n-1)nab}{2}}.$$

In particular

$$(x^a y^b z^{-ab/2})^n = x^{na} y^{nb} z^{-\frac{n^2 ab}{2}}.$$

It follows that the cyclic group $\langle x^a y^b z^{-ab/2}\rangle$ depends only on the point of the projective line over $\mathbf{F}_p$ whose homogeneous coordinates are $[a : b]$. Letting $[a : b]$ run over the $p+1$ points of that projective line, we obtain $p+1$ cyclic subgroups of $H(\mathbf{F}_p)$, namely $\{\langle x\rangle\} \cup \{\langle x^a yz^{-\frac{a}{2}}\rangle \mid a = 0, \dots p-1\}$. Together with the center of $H(\mathbf{F}_p)$, namely the cyclic group $\langle z\rangle$, we obtain a family of $p+2$ cyclic subgroups. Routine calculations show that the conditions from Proposition 7 are satisfied with $s = p-1$ and $t = p+1$. □

Specializing to the case $p = 7$, we obtain:

Corollary 9. *The link at the vertex b in the local development of the complex $E(\mathcal{Y})$ around a is isomorphic, as a simplicial graph, to the incidence graph of a generalized quadrangle of order $(6, 8)$. In particular it has diameter 4 and girth 8.*

Proof. This follows directly from Lemma 2 and Corollary 8. □

5. Developability of $E(\mathcal{Y})$

We are now able to complete the proof that the group $E \cong \widehat{E(\mathcal{Y})}$ is infinite hyperbolic.

Proposition 10. *The group E is infinite hyperbolic; it acts properly cocompactly by isometric automorphisms on the global development $|D(\mathcal{Y})|$, which is a CAT(-1) space. That space is geodesically complete: every geodesic segment can be extended to a geodesic line. Furthermore, every finite subgroup of E is conjugate to a subgroup of E_a, E_b or E_{c_i} for some i, and E is virtually torsion-free.*

Proof. Since the affine realization $|\mathcal{Y}|$ is simply connected and endowed the structure of a finite piecewise real hyperbolic triangle complex, all we must do to prove that $E(\mathcal{Y})$ is developable is to show that the link of every vertex $\tau \in \mathcal{Y}$ in the local development of $E(\mathcal{Y})$ around τ, is CAT(1) (see [BH, Remark II.12.27(2)]). This is the so called *Link Condition*. Since $\mathcal{Y}$ is 2-dimensional, it suffices to prove that every injective loop in the link has length at least 2π (see [BH, §II.5.24]). We consider the vertices a, b and c_i successively.

- The angle at a in Δ is $\pi/6$. Moreover, by Corollary 5, the link at a in the local development of $E(\mathcal{Y})$ around a has girth 12. We conclude that every injective loop in that link has length at least 2π, as required.
- The angle at b in Δ is $\pi/4$. Moreover, by Corollary 9, the link at b in the local development of $E(\mathcal{Y})$ around b has girth 8. We conclude that the Link Condition is also satisfied at b.
- The link at c_i is the complete bipartite graph $K_{2,7}$, its girth is 4, so the Link Condition is satisfied at c_i for every i.

The developability of $E(\mathcal{Y})$ therefore follows from [BH, Theorem II.12.28], which also ensures that $|D(\mathcal{Y})|$ is a CAT(-1) space. Since E acts properly and cocompactly on $|D(\mathcal{Y})|$, it follows that E is Gromov hyperbolic.

Observe that the link of every vertex in the local developments of $E(\mathcal{Y})$ has the following property: given a point p in that link, there is a point q at distance π from p. This implies that every geodesic segment in the development $|D(\mathcal{Y})|$ can be prolonged locally beyond its extremities. It follows that $|D(\mathcal{Y})|$ is geodesically complete. In particular it is unbounded. Hence E is infinite.

That every finite subgroup of E is conjugate to a subgroup of a vertex group also follows from [BH, Theorem II.12.28]. In order to prove that E is virtually torsion-free, we observe from the presentation of E that this group has two retractions $E \to E_a$ and $E \to E_b$. Their product $\rho\colon E \to E_a \times E_b$ yields a finite quotient isomorphic to $D_{146} \times H(\mathbf{F}_7)$; moreover, for each i, the vertex group E_{c_i} splits as a direct product $E_{c_i} \cong E_{ac_i} \times E_{bc_i}$. Therefore, any non-trivial element of a vertex group has a non-trivial image under $\rho\colon E \to E_a \times E_b$. It follows that $\mathrm{Ker}(\rho)$ is a finite index torsion-free subgroup of E. □

6. A hyperbolic triangle building

A *hyperbolic triangle building* of type (p,q,r) is a 2-dimensional building whose type is given by the hyperbolic Coxeter group generated by the reflections across the sides of a geodesic triangle with angles π/p, π/q and π/r in the hyperbolic plane. Our next goal is to observe that the global development $D(\mathcal{Y})$ is a hyperbolic triangle building of type $(2,4,6)$.

What we did so far provides a detailed description of the link of every vertex in the global development $D(\mathcal{Y})$. Indeed, we have seen that every such link is a copy of the complete bipartite graph $K_{2,7}$ (for the vertices in the orbit of c_i for some i), or the first barycentric subdivision of the incidence graph of a projective plane (for vertices in the orbit of a), or the incidence graph of a generalized quadrangle. All these graphs are special instances of 1-dimensional spherical buildings. We are thus in a position to invoke a theorem of Jacques Tits, providing a local characterization of buildings (see [Dav, Chapter 18] for generalities on geometric realizations of buildings).

Proposition 11. *The global development* $D(\mathcal{Y})$ *is a hyperbolic triangle building of type* $(2,4,6)$. *The natural* E*-action on that building is type-preserving.*

Proof. Given the shape of the link of every vertex, described by Lemma 2 and Corollaries 5 and 9, the required conclusion follows from [Tit, Theorem 1]. □

In the building on which E acts naturally, the edges covering ab form a single E-orbit; since the edge group E_{ab} is trivial, the action of E on that orbit is moreover free. Thus, by construction, the E-action on the associated building is sharply transitive on the panels of a certain type. A study of 'short presentations' for groups acting sharply transitively on panels of one type in 2-dimensional Euclidean buildings was performed by Jan Essert in [Ess]; in some sense, the present paper provides a hyperbolic analogue of that study. For that reason, the group appearing in Theorem 1 is denoted by the letter E.

7. The spectral criterion for Property (T)

Our final task to complete the proof of Theorem 1 consists in checking that E has Kazhdan's property (T). To that end, we rely on a result due to Izhar Oppenheim from [Opp]. It provides a sufficient condition for a group acting on a 2-dimensional simplicial complex X to enjoy Kazhdan's property (T), provided the smallest eigenvalue of the Laplace operator on the links of vertices of X satisfy a suitable condition. In the case of the complex $D(\mathcal{Y})$, we already know the shape of the link of every vertex: it is a complete bipartite graph, or the first barycentric subdivision of the indicidence graph of a projective plane of order 8, or the incidence graph of a generalized quadrangle of order $(6,8)$. For each of these graphs, the full spectrum of the Laplace operator is known: it can be extracted from the work of Feit–Higman [FH]. However, a calculation shows that the hypotheses of [Opp, Theorem 1] are not satisfied by the complex $D(\mathcal{Y})$.

In order to overcome that difficulty, we use the following trick. In the simplicial complex $D(\mathcal{Y})$, the edges in the E-orbit of $[ac_i]$ for some i are characterized by the property that they are the only edges contained in exactly two 2-simplices. Such edges are called *thin*. Let $[\bar{a}\bar{c_i}]$ be such an edge. Let also $\bar{b}$ and $\bar{b}'$ the only two vertices (both in the E-orbit of b) such that $\bar{a}\bar{b}\bar{c_i}$ and $\bar{a}\bar{b}'\bar{c_i}$ are both the vertex sets of a 2-simplex. For all thin edges of the form $[\bar{a}\bar{c_i}]$, we replace the subcomplex spanned by $\bar{a}, \bar{b}, \bar{b}', \bar{c_i}$ by a subcomplex containing a single 2-simplex, spanned by $\bar{a}, \bar{b}, \bar{b}'$ (see Figure 3). The new complex obtained in this way is denoted by X. The operation of replacing $D(\mathcal{Y})$ by X is purely combinatorial, it does not affect the metric on $|D(\mathcal{Y})|$. Moreover the E-action on $D(\mathcal{Y})$ canonically determines an action on X by isometric automorphisms. Every 2-simplex in the metric realization $|X|$ is now isometric to a geodesic triangle with angles $\pi/3, \pi/4, \pi/4$ in the real hyperbolic plane.

The operation of replacing $D(\mathcal{Y})$ by X does not modify the shape of the links at vertices $\bar{b} \in X$ in the E-orbit of b. On the other hand, if $\bar{a}$ is a vertex

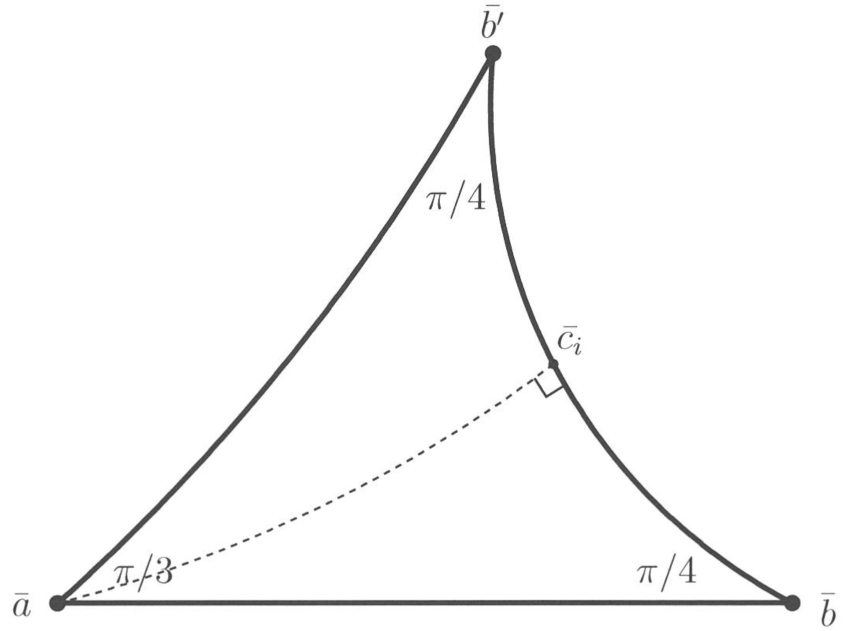

FIGURE 3
Discarding the thin edges

in the E-orbit of a, then the link of X at $\bar{a}$ is now the incidence graph of a projective plane of order 8 and no longer its first barycentric subdivision. Indeed, the vertices of degree 2 in that subdivision are discarded, since they correspond to the thin edges in $D(\mathcal{Y})$.

We now state a criterion for Property (T) which follows easily from the main result of [Opp].

Proposition 12. *Let X be a 2-dimensional simplicial complex and Γ be a discrete group acting properly, cocompactly on X. Let $p \geq 3$ be an integer. Assume that in every 2-simplex of X, the link of one vertex in X is isomorphic to the incidence graph of a projective plane of order $p+1$, and the link of the other two vertices is isomorphic to the incidence graph of a generalized quadrangle of order $(p-1, p+1)$. If $p \geq 6$, then Γ has Kazdhan's Property (T).*

Proof. We need to know the smallest positive eigenvalue of the Laplace operator on the incidence graph of a projective plane of order q (resp. a generalized quadrangle of order (s,t)). The spectrum of that operator can be extracted from the computations made in [FH, Lemmas 3.3 and 5.1], but the corresponding result is not stated explicitly in that reference. An explicit computation of the spectrum is done in [Gar, Prop. 7.10] under the extra hypothesis that the generalized polygon is associated to a group with a BN-pair. However, that extra hypothesis is not needed (see for example [BdlHV, Proposition 5.7.6] for the case of projective planes). The result is that the smallest positive eigenvalue of the Laplacian of the incidence graph of a projective plane of order q (resp. a generalized quadrangle of order (s,t)) is $1 - \frac{\sqrt{q}}{q+1}$ (resp. $1 - \sqrt{\frac{s+t}{(1+s)(1+t)}}$). Taking $q = p+1$ (resp. $(s,t) = (p-1, p+1)$), we find $\lambda_P = 1 - \frac{\sqrt{p+1}}{p+2}$ (resp. $\lambda_Q = 1 - \sqrt{\frac{2}{p+2}}$). By [Opp, Theorem 1], the group Γ has Property (T) provided that the following two conditions hold:

- $\lambda_P + 2\lambda_Q > 3/2$,
- $(\lambda_P + \lambda_Q - 1)^2 + 2(\lambda_P + \lambda_Q - 1)(2\lambda_Q - 1) > 0$.

A straightforward computation shows that the first condition holds for all integer $p \geq 5$, while the second holds for all $p \geq 6$. □

End of the proof of Theorem 1. That E is infinite hyperbolic and virtually torsion-free follows from Proposition 10. The discussion at the beginning of the present section shows that the 2-complex $D(\mathcal{Y})$ can be replaced, in a canonical way, by a 2-complex X satisfying the hypotheses of Proposition 12. The latter shows that E has Property (T).

Invoking [Tit, Theorem 1], we infer that the simplicial complex X is a hyperbolic triangle building of type $(3,4,4)$ which is *thick* (i.e., it contains no thin edge). This is global feature of X allows us to invoke [Xie], which ensures that E is quasi-isometrically rigid. □

8. Variations on the same theme

There is a certain amount of flexibility in the construction of the group E which can be exploited to provide many more infinite hyperbolic Kazhdan groups similar to E. The vertex groups E_{c_i} need not be cyclic: they could also be chosen to be the dihedral group D_{14} of order 14. One could also permute the edge groups E_{ac_i} arbitrarily without changing E_{bc_i}. The specific choice for the group E in Theorem 1 was made in order to minimize the maximal length of a relation.

Let us note that one can also obtain larger siblings of E as follows. For any Mersenne prime p, define a simple complex of groups consisting of $p+2$ hyperbolic triangles of type $(2,4,6)$ glued along their hypothenuse. The two acute vertex groups are a Heisenberg group over $\mathbf{F}_p$ and a dihedral group D_{2n} of order $2n$, where $n=(p+1)^2+p+2$, respectively. The other $p+2$ vertex groups are cyclic or dihedral of order $2p$. The edge groups are chosen using Proposition 4 and Corollary 8 so that the Link Condition is satisfied at every vertex. We need p to be a Mersenne prime since $p+1$ must be a prime power to guarantee that the hypotheses of Proposition 4 are fulfilled, see Remark 6. The fundamental group of that complex is always hyperbolic, and it has Property (T) for all $p \geq 7$ by Proposition 12.

For the Mersenne prime $p=3$, using the difference set $\mathcal{D}=\{0,1,4,14,16\}=\{0,1,4,-7,-5\}$ in the cyclic group $\mathbf{Z}/21\mathbf{Z}$, we obtain the following group presentation:

$$\begin{aligned} E_3 = \big\langle x,y,z,t,r \mid\ & x^3, y^3, [x,y]z^{-1}, [x,z], [y,z], \\ & t^2, r^{21}, trtr, \\ & [xyz,t], [x^{-1}yz^{-1}, tr], \\ & [x, tr^4], [y, tr^{-7}], [z, tr^{-5}]\big\rangle. \end{aligned}$$

After substituting $z=[x,y]$, we obtain a 12-relator presentation for E_3 in which all relators have length ≤ 21. The same arguments as for E show that E_3 is infinite, hyperbolic, virtually torsion-free and that it acts geometrically on a thick hyperbolic triangle building of type $(3,4,4)$. However, the spectral criterion for Property (T) from Proposition 12 does not apply, and the following question remains open:

Question 13. Does the group E_3 have Kazhdan's property (T)?

That question might be approached using similar methods as in [KNO].

We finish this note by recording another observation that follows from combining Proposition 4 with Marc Bourdon's construction from [Bou, §1.5.3] and its extension due to Jacek Świątkowski [Świ].

Proposition 14. *Let L be the incidence graph of a finite generalized n-gon of order (s,t) with $n \geq 3$ (i.e., a bipartite graph of diameter n and girth $2n$ such that every vertex has degree $s+1$ or $t+1$). Assume that t is a prime power.*

Then there is a group Γ acting faithfully, properly and cocompactly (but not type preservingly) on a thick locally finite triangle building X of type $(3,n,n)$ admitting L as the link of a vertex.

Proof. We follow the construction described in [Świ, §5.3] in order to build Γ as the fundamental group of a simple complex of finite groups $\Gamma = \widehat{G(\mathcal{Y})}$. The underlying complex $\mathcal{Y}$ is the simplicial cone over the graph L. Let $V = V_1 \cup V_2$ be the bipartition of the vertex set of L, so that every edge in L joins a vertex in V_1 to a vertex in V_2, every vertex in V_1 has degree $s+1$ and every vertex in V_2 has degree $t+1$. To each vertex v in V_2, we define the vertex group G_v as a dihedral group of order $2(t^2+t+1)$. To each edge e belonging to the set $E_L(v)$ of edges of L emanating from v, we define G_e as a cyclic group of order 2. For all $e \in E_L(v)$ we define the inclusion of G_e into G_v in such a way that the bipartite coset graph of G_v with respect to $\{G_e \mid e \in E_L(v)\}$ is the first barycentric subdivision of the incidence graph of the Desarguesian projective plane of order t. Such a choice is possible in view of Proposition 4 and Remark 6; this is where we use the hypothesis that t is a prime power. For $v \in V_1$ we define the vertex group G_v to be cyclic of order 2, and identify G_v with all edge groups G_e with $e \in E_L(v)$. The groups attached to all the other simplices of $\mathcal{Y}$ are trivial. By [BH, Theorem II.12.28], the simple complex of groups $G(\mathcal{Y})$ defined in this way is developable. By [Tit, Theorem 1], the development $D(\mathcal{Y})$ is a non-thick triangle building of type $(2,6,n)$. Upon discarding the edges of $D(\mathcal{Y})$ that cover edges of L, we may view $D(\mathcal{Y})$ is a thick triangle building of type $(3,n,n)$ on which $\Gamma = \widehat{G(\mathcal{Y})}$ acts faithfully, properly and cocompactly, but not type-preservingly. □

The difference between Bourdon's construction [Bou, §1.5.3] and Proposition 14 is that the former yields triangle buildings of type $(2,n,n)$.

Remark 15. Proposition 14 comes close to a solution of a problem posed by W. Kantor [Kan2, Problem C.6.7]. It notably implies that all finite projective planes satisfying the Prime Power Conjecture appear as residue planes in $\tilde{A}_2$-buildings admitting a discrete cocompact group of automorphisms. In particular, all known non-Desarguesian finite projective planes do. This provides a construction of an infinite family of cocompact lattices in *exotic* $\tilde{A}_2$-buildings of arbitrarily large thickness, where *exotic* means non-isomorphic to the Bruhat–Tits building of a simple algebraic group over a local field. In particular, the main result of [BCL] applies to those lattices, which ensures that they do not admit any finite-dimensional representation with infinite image over any field. The first construction of an infinite family of cocompact lattices in exotic $\tilde{A}_2$-buildings was obtained in [BCL, §10]; since then another source of cocompact lattices in exotic $\tilde{A}_2$-buildings of arbitrarily large thickness has been identified by N. Radu [Rad1]. The first example of a cocompact lattice in an $\tilde{A}_2$-building admitting non-Desarguesian residue planes is due to him [Rad2]. That example remains the only known $\tilde{A}_2$-building with a cocompact lattice where *all* residue planes are non-Desarguesian.

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