# Berkovich spaces embed in Euclidean spaces

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# Berkovich spaces embed in Euclidean spaces

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**Abstract.** Let $K$ be a field that is complete with respect to a nonarchimedean absolute value such that $K$ has a countable dense subset. We prove that the Berkovich analytification $V^{\mathrm{an}}$ of any $d$-dimensional quasi-projective scheme $V$ over $K$ embeds in $\mathbb{R}^{2d+1}$. If, moreover, the value group of $K$ is dense in $\mathbb{R}_{>0}$ and $V$ is a curve, then we describe the homeomorphism type of $V^{\mathrm{an}}$ by using the theory of local dendrites.

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## 1. Introduction

In this article, *valued field* will mean a field $K$ equipped with a nonarchimedean absolute value $|\ |$ (or equivalently with a valuation taking values in an additive subgroup of $\mathbb{R}$). Let $K$ be a complete valued field. Let $V$ be a quasi-projective $K$-scheme. The associated Berkovich space $V^{\mathrm{an}}$ [Be1, §3.4] is a topological space that serves as a nonarchimedean analogue of the complex analytic space associated to a complex variety. (Actually, $V^{\mathrm{an}}$ carries more structure, but it is only the underlying topological space that concerns us here.) Although the set $V(K)$ in its natural topology is totally disconnected, $V^{\mathrm{an}}$ is arcwise connected if and only if $V$ is connected; moreover, the topological dimension of $V^{\mathrm{an}}$ equals the dimension of the scheme $V$ [Be1, Theorems 3.4.8(iii,iv) and 3.5.3(iii,iv)]. Also, $V^{\mathrm{an}}$ is locally contractible: see [Be3, Be4] for the smooth case, and [HL, Theorem 13.4.1] for the general case. Although Berkovich spaces are not always metrizable, they retain certain countability features in general; cf. [Fa] and [Poi].

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Our goal is to study the topology of $V^{\mathrm{an}}$ under a countability hypothesis on $K$ with its absolute value topology. For instance, we prove the following:

**Theorem 1.1.** *Let $K$ be a complete valued field having a countable dense subset. Let $V$ be a quasi-projective $K$-scheme of dimension $d$. Then $V^{\mathrm{an}}$ is homeomorphic to a topological subspace of $\mathbb{R}^{2d+1}$.*

**Remark 1.2.** The hypothesis that $K$ has a countable dense subset is necessary as well as sufficient. Namely, $K$ embeds in $(\mathbb{A}^1_K)^{\mathrm{an}}$, so if the latter embeds in a separable metric space such as $\mathbb{R}^n$, then $K$ must have a countable dense subset.

**Remark 1.3.** The hypothesis is satisfied when $K = \mathbb{Q}_p$ or $\mathbb{F}_p((t))$. It is satisfied also when $K$ is the completion of an algebraic closure of a completion of a global field $k$, i.e., when $K$ is $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$ or its characteristic $p$ analogue $\widehat{\overline{\mathbb{F}_p((t))}}$, because the algebraic closure of $k$ in $K$ is countable and dense. It follows that the hypothesis is satisfied also for any complete subfield of these two fields.

Recall that a valued field is called *spherically complete* if every descending sequence of balls has nonempty intersection. Say that $K$ has *dense value group* if $|\ |\colon K^\times \to \mathbb{R}_{>0}$ has dense image, or equivalently if the value group is not isomorphic to $\{0\}$ or $\mathbb{Z}$.

**Remark 1.4.** The separability hypothesis fails for any spherically complete field $K$ with dense value group. Proof: Let $(t_i)$ be a sequence of elements of $K$ such that the sequence $|t_i|$ is strictly decreasing with positive limit. For each sequence $\epsilon = (\epsilon_i)$ with $\epsilon_i \in \{0, 1\}$, define

$$U_\epsilon := \left\{ x \in K : \left| x - \textstyle\sum_{i=1}^n \epsilon_i t_i \right| < |t_n| \text{ for all } n \right\}.$$

The $U_\epsilon$ are uncountably many disjoint open subsets of $K$, and each is nonempty by definition of spherically complete.

Let us sketch the proof of Theorem 1.1. We may assume that $V$ is projective. The key is a result that presents $V^{\mathrm{an}}$ as a filtered limit of finite simplicial complexes. Variants of this limit description have appeared in several places in the literature (see the end of [Pa, Section 1] for a summary); for convenience, we use [HL, Theorem 13.2.4], a version that does not assume that $K$ is algebraically closed (and that proves more than we need, namely that the maps in the inverse limit can be taken to be strong deformation retractions). Our hypothesis on $K$ is used to show that the index set for the limit has a countable cofinal subset. To complete the proof, we use a well-known result from topology, Proposition 3.1, that an inverse limit of a sequence of finite simplicial complexes of dimension at most $d$ can be embedded in $\mathbb{R}^{2d+1}$.

**Remark 1.5.** If we wanted to prove only that the space $V^{\mathrm{an}}$ in Theorem 1.1 is metrizable, we could avoid the use of [HL, Theorem 13.2.4], and instead simply use the Urysohn metrization theorem, as we now explain. Let $K_0$ be a countable dense subset of $K$. Let $A$ be the (countable) set of polynomials in $K[x_1, \dots, x_n]$ whose nonzero coefficients lie in $K_0$. Suppose that $D$ is a Berkovich $n$-dimensional polydisk. For each $a \in A$, let $r_a$ be an upper bound for $a$ on $D$. The map sending a seminorm on $K[x_1, \dots, x_n]$ to its values on $A$ embeds $D$ in the space $\prod_{a \in A}[0, r_a]$ with the product topology, and the latter is second countable, so $D$ is second countable. Next, $(\mathbb{A}^n_K)^{\mathrm{an}}$ is a countable union of such polydisks $D$, and for any affine variety $V_0$ the space $V_0^{\mathrm{an}}$ is a subspace of some $(\mathbb{A}^n_K)^{\mathrm{an}}$, and for any finite-type $K$-scheme $V$, the space $V^{\mathrm{an}}$ is a finite union of such spaces $V_0^{\mathrm{an}}$, so all of these are second countable. If $V$ is a proper $K$-scheme, then $V^{\mathrm{an}}$ is also compact and Hausdorff [Bel, Theorems 3.4.8(ii) and 3.5.3(ii)], so the Urysohn metrization theorem applies to $V^{\mathrm{an}}$. More generally, if $V$ is any separated finite-type $K$-scheme, Nagata's compactification theorem [Nag] (see [Lü, Co] for modern treatments) provides an open immersion of $V$ into a proper $K$-scheme $\overline{V}$, and then $V^{\mathrm{an}}$ is a subspace of $\overline{V}^{\mathrm{an}}$, so $V^{\mathrm{an}}$ is metrizable again.

**Remark 1.6.** Although $V^{\mathrm{an}}$ is metrizable, it typically has no *canonical* metric. To be precise, if $K$ is nondiscrete, there is no metric on $(\mathbb{P}^1_K)^{\mathrm{an}}$ that is $\mathrm{Aut}(\mathbb{P}^1_K)$-invariant. This is because $\mathrm{Aut}(\mathbb{P}^1_K)$ acts transitively on pairs of points of $\mathbb{P}^1(K)$, so all distances would have be the same, contradicting the fact that the subspace topology on $\mathbb{P}^1(K)$ induced from $(\mathbb{P}^1_K)^{\mathrm{an}}$ is the usual, nondiscrete one. See Remark 8.7, however.

**Remark 1.7.** It seems likely also that Theorem 1.1 holds for any separated finite-type $K$-scheme $V$ of dimension $d$.

Our article is organized as follows. Sections 2 and 3 give a quick proof of Proposition 3.1. Section 4 proves a result needed to replace $K$ by a countable subfield, in order to obtain a countable index set for the inverse limit. Section 5 combines all of the above to prove Theorem 1.1. The final sections of the paper study the topology of Berkovich curves: after reviewing and developing the theory of dendrites and local dendrites in Sections 6 and 7, respectively, we show in Section 8 how to obtain the homeomorphism type of any Berkovich curve over $K$ as above. For example, as a special case of Corollary 8.2, we show that $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{an}}$

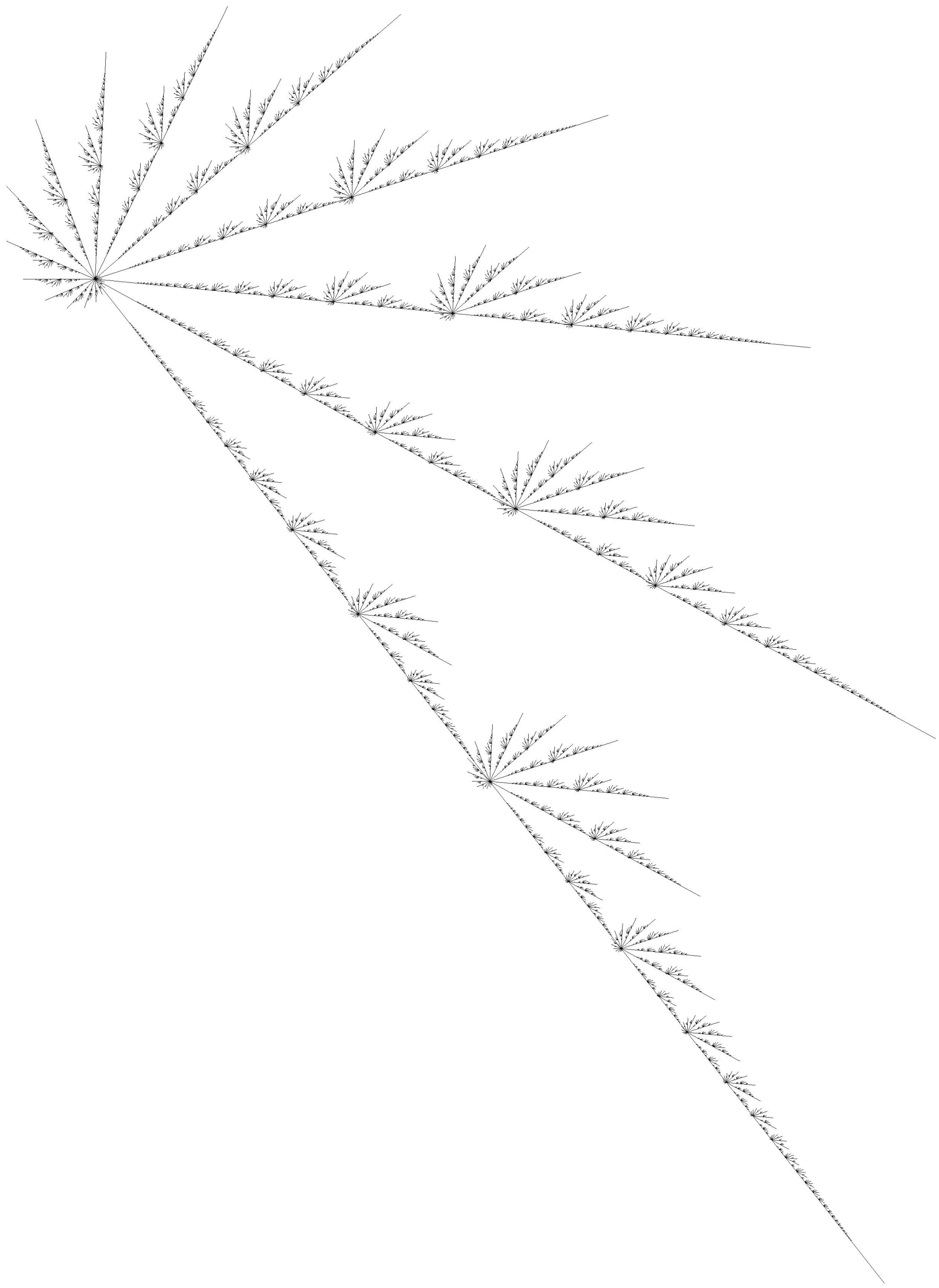

Figure 1
The Berkovich projective line over $\mathbb{C}_p$, also known as the Ważewski universal dendrite

is homeomorphic to a topological space first constructed in 1923, the Ważewski universal dendrite [Wa], depicted in Figure 1.[1]

[1] We believe that ours is the first topologically accurate depiction of $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{an}}$ in the literature: to obtain the correct topology, the branches emanating from each branch point must have diameters tending to $0$. In our depiction, all branches (including branches of ... of branches) are similar; but eventually, at a scale too small to see on the page, they must cease to meet at equal angles and their diameters should decrease faster than geometrically, in order to avoid unwanted intersections.

## 2. Approximating maps of finite simplicial complexes by embeddings

If $X$ is a topological space, a map $f: X \to \mathbb{R}^n$ is called an *embedding* if $f$ is a homeomorphism onto its image. For compact $X$, it is equivalent to require that $f$ be a continuous injection. When we speak of a finite simplicial complex, we always mean its geometric realization, a compact subset of some $\mathbb{R}^n$. A set of points in $\mathbb{R}^n$ is said to be in *general position* if for each $m \le n-1$, no $m+2$ of the points lie in an $m$-dimensional affine subspace.

**Lemma 2.1.** *Let $X$ be a finite simplicial complex of dimension at most $d$. Let $\epsilon \in \mathbb{R}_{>0}$. For any continuous map $f: X \to \mathbb{R}^{2d+1}$, there is an embedding $g: X \to \mathbb{R}^{2d+1}$ such that $|g(x) - f(x)| \le \epsilon$ for all $x \in X$.*

*Proof.* The simplicial approximation theorem implies that $f$ can be approximated within $\epsilon/2$ by a piecewise linear map $g_0$. For each vertex $x_i$ in the corresponding subdivision of $X$, in turn, choose $y_i \in \mathbb{R}^{2d+1}$ within $\epsilon/2$ of $g_0(x_i)$ so that the $y_i$ are in general position. Let $g: X \to \mathbb{R}^{2d+1}$ be the piecewise linear map, for the same subdivision, such that $g(x_i) = y_i$. Then $g$ is injective, and $g$ is within $\epsilon/2$ of $g_0$, so $g$ is within $\epsilon$ of $f$. $\square$

## 3. Inverse limits of finite simplicial complexes

**Proposition 3.1.** *Let $(X_n)_{n\ge 0}$ be an inverse system of finite simplicial complexes of dimension at most $d$ with respect to continuous maps $p_n: X_{n+1} \to X_n$. Then the inverse limit $X := \varprojlim X_n$ embeds in $\mathbb{R}^{2d+1}$.*

*Proof.* For $m \ge 0$, let $\Delta_m \subseteq X_m \times X_m$ be the diagonal, and write $(X_m \times X_m) - \Delta_m = \bigcup_{n=m}^{\infty} C_{mn}$ with $C_{mn}$ compact. For $0 \le m \le n$, let $D_{mn}$ be the inverse image of $C_{mn}$ in $X_n \times X_n$. Let $K_n = \bigcup_{m=1}^{n} D_{mn}$. Since $K_n$ is closed in $X_n \times X_n$, it is compact.

For $n \ge 0$, we inductively construct an embedding $f_n: X_n \to \mathbb{R}^{2d+1}$ and numbers $\alpha_n, \epsilon_n \in \mathbb{R}_{>0}$ such that the following hold for all $n \ge 0$:

(i) If $(x, x') \in K_n$, then $|f_n(x) - f_n(x')| \ge \alpha_n$.

(ii) $\epsilon_n < \alpha_n/4$.

(iii) $\epsilon_n < \epsilon_{n-1}/2$ (if $n \ge 1$).

(iv) If $x \in X_{n+1}$, then $|f_{n+1}(x) - f_n(p_n(x))| \le \epsilon_n$.

Let $f_0: X_0 \to \mathbb{R}^{2d+1}$ be any embedding (apply Lemma 2.1 to a constant map, for instance). Now suppose that $n \ge 0$ and that $f_n$ has been constructed. Since $f_n$ is injective and $K_n$ is compact, we may choose $\alpha_n \in \mathbb{R}_{>0}$ satisfying (i).

Choose any $\epsilon_n \in \mathbb{R}_{>0}$ satisfying (ii) and (iii). Apply Lemma 2.1 to $p_n \circ f_n$ to find $f_{n+1}$ satisfying (iv). This completes the inductive construction.

Now $\sum_{i=n}^{\infty} \epsilon_i < 2\epsilon_n < \alpha_n/2$ by (iii) and (ii). Let $\widehat{f}_n$ be the composition $X \to X_n \xrightarrow{f_n} \mathbb{R}^{2d+1}$. For $x \in X$, (iv) implies $|\widehat{f}_{n+1}(x) - \widehat{f}_n(x)| \le \epsilon_n$, so the maps $\widehat{f}_n$ converge uniformly to a continuous map $f \colon X \to \mathbb{R}^{2d+1}$ satisfying $|f(x) - f_n(x_n)| < \alpha_n/2$.

We claim that $f$ is injective. Suppose that $x = (x_n)$ and $x' = (x'_n)$ are distinct points of $X$. Fix $m$ such that $x_m \ne x'_m$. Fix $n \ge m$ such that $(x_m, x'_m) \in C_{mn}$. Then $(x_n, x'_n) \in D_{mn} \subseteq K_n$. By (i), $|f_n(x_n) - f_n(x'_n)| \ge \alpha_n$. On the other hand, $|f(x) - f_n(x_n)| < \alpha_n/2$ and $|f(x') - f_n(x'_n)| < \alpha_n/2$, so $f(x) \ne f(x')$. □

**Remark 3.2.** Proposition 3.1 was proved in the 1930s. Namely, following a 1928 sketch by Menger, in 1931 it was proved independently by Lefschetz [Le], Nöbeling [Nö], and Pontryagin and Tolstowa [PT] that any compact metrizable space of dimension at most $d$ embeds in $\mathbb{R}^{2d+1}$. The proofs proceed by using Alexandroff's idea of approximating compact spaces by finite simplicial complexes (nerves of finite covers), so even if it not obvious that the 1931 *result* applies directly to an inverse limit of finite simplicial complexes of dimension at most $d$ (i.e., whether such an inverse limit is of dimension at most $d$), the *proofs* still apply. And in any case, in 1937 Freudenthal [Fr] proved that a compact metrizable space is of dimension at most $d$ if and only if it is an inverse limit of finite simplicial complexes of dimension at most $d$. See Sections 1.11 and 1.13 of [En] for more about the history, including later improvements.

## 4. Berkovich spaces over noncomplete fields

Berkovich analytifications were originally defined only when the valued field $K$ was complete [Be1, Sections 3.4 and 3.5]. For a quasi-projective variety $V$ over an *arbitrary* valued field $K$, there are two approaches to defining the topological space $V^{\mathrm{an}}$:

1. Use the same definition as for complete fields in [Be1], in terms of seminorms.
2. Use a definition as in [HL, Section 13.1] in terms of types over $K \cup \mathbb{R}$.

As shown in [HL, Section 13.1], these two definitions yield homeomorphic topological spaces when $K$ is complete. One advantage of the second definition is that it can be used in more general situations, for fields with a valuation whose value group is not contained in $\mathbb{R}$. But given the aims of this paper, we will use the first definition from now on.

The following proposition shows that no new spaces arise by allowing noncomplete fields: it would have been equivalent to define $V^{\mathrm{an}}$ as $(V_{\widehat{K}})^{\mathrm{an}}$ (the subscript denotes base extension).

**Proposition 4.1.** *Let $K \leq L$ be an extension of valued fields such that $K$ is dense in $L$. Let $V$ be a quasi-projective $K$-variety. Then $(V_L)^{\mathrm{an}}$ is naturally homeomorphic to $V^{\mathrm{an}}$.*

*Proof.* This follows by tracing through the construction of $V^{\mathrm{an}}$ in [Be1, Sections 3.4 and 3.5]. The key point is that each multiplicative seminorm on $K[t_1, \dots, t_n]$ is the restriction of a unique multiplicative seminorm on $L[t_1, \dots, t_n]$, obtained as the unique continuous extension. □

**Remark 4.2.** Proposition 4.1 can be proved also for the second definition, in terms of types not extending the value group, even for fields with value group not contained in $\mathbb{R}$; the restriction map remains bijective. This shows that the two definitions produce homeomorphic topological spaces for any valued field $K$ with value group contained in $\mathbb{R}$, even when $K$ is not complete.

## 5. Embeddings of Berkovich spaces

**Proposition 5.1.** *Let $K$ be a valued field having a countable dense subset. Let $V$ be a* projective *$K$-scheme of dimension $d$. Then $V^{\mathrm{an}}$ is homeomorphic to an inverse limit $\varprojlim X_n$ where each $X_n$ is a finite simplicial complex of dimension at most $d$ and each map $X_{n+1} \to X_n$ is continuous.*

*Proof.* First suppose that $K$ is countable. Since $V$ is projective, $V^{\mathrm{an}}$ is compact, so we may apply [HL, Theorem 13.2.4] to $V^{\mathrm{an}}$ to obtain that $V^{\mathrm{an}}$ is a filtered limit of finite simplicial complexes over an index set $I$. Since $K$ is countable, the proof of [HL, Theorem 13.2.4] shows that $I$ may be taken to be countable, so our limit may be taken over a sequence, as desired.

Now assume only that $K$ has a countable dense subset. Since $V$ is of finite presentation over $K$, it is the base extension of a projective scheme $V_0$ over a countable subfield $K_0$ of $K$. By adjoining to $K_0$ a countable dense subset of $K$, we may assume that $K_0$ is dense in $K$. By Proposition 4.1, $V^{\mathrm{an}}$ is homeomorphic to $(V_0)^{\mathrm{an}}$, which has already been shown to be an inverse limit of the desired form. □

**Proposition 5.2.** *Let $K$ be a complete valued field. If $U$ is an open subscheme of $V$, then the induced map $U^{\mathrm{an}} \to V^{\mathrm{an}}$ is a homeomorphism onto an open subspace.*

*Proof.* This is a consequence of the construction of $V^{\mathrm{an}}$ by gluing the analytification of affine open subschemes of $V$: see step (2) in the proof of [Be1, Theorem 3.4.1], and see [Be1, Proposition 3.4.6(8)] for the statement itself; in that section, the valuation on $K$ is assumed to be nontrivial, but as remarked in the first sentence of the proof of [Be1, Theorem 3.5.1], the same argument works when the valuation is trivial. □

Theorem 1.1 follows immediately from Propositions 3.1, 5.1, and 5.2.

## 6. Dendrites

When $V$ is a curve, more can be said about $V^{\mathrm{an}}$. But first we recall some definitions and facts from topology.

**6.1. Definitions.** A *continuum* is a compact connected metrizable space (the empty space is not connected). A *simple closed curve* in a topological space is any subspace homeomorphic to a circle. A *dendrite* is a locally connected continuum containing no simple closed curve. Dendrites may be thought of as topological generalizations of trees in which branching may occur at a dense set of points. A point $x$ in a dendrite $X$ is called a *branch point* if $X-\{x\}$ has three or more connected components; these components are then called the *branches* at $x$.

**6.2. Ważewski's theorems.** The following three theorems were proved by Ważewski in his thesis [Wa].[2]

**Theorem 6.1.** *Up to homeomorphism, there is a unique dendrite $W$ such that its branch points are dense in $W$ and there are $\aleph_0$ branches at each branch point.*

The dendrite $W$ in Theorem 6.1 is called the *Ważewski universal dendrite*.

**Theorem 6.2.** *Every dendrite embeds in $W$.*

[2] Actually, Ważewski used a different, equivalent definition: for him, a dendrite was any image $D$ of a continuous map $[0,1] \to \mathbb{R}^n$ such that $D$ contains no simple closed curve. A dendrite in Ważewski's sense is a dendrite in our sense by [Nad, Corollary 8.17]. Conversely, a dendrite in our sense embeds in $\mathbb{R}^2$ by [Nad, Section 10.37] (or, alternatively, is an inverse limit of finite trees by [Nad, Theorem 10.27] and hence embeds in $\mathbb{R}^3$ by Proposition 3.1), and is a continuous image of $[0,1]$ by the Hahn–Mazurkiewicz theorem [Nad, Theorem 8.14].

**Theorem 6.3.** *Every dendrite is homeomorphic to the image of some continuous map* $[0,1] \to \mathbb{R}^2$.

**6.3. Pointed dendrites.** A *pointed dendrite* is a pair $(X, P)$ where $X$ is a dendrite and $P \in X$. An *embedding of pointed dendrites* is an embedding of topological spaces mapping the point in the first to the point in the second. Let $\mathcal{P}$ be the category of pointed dendrites, in which morphisms are embeddings. By the *universal pointed dendrite*, we mean $W$ equipped with one of its branch points $w$.

**Theorem 6.4.** *Every pointed dendrite* $(X, P)$ *admits an embedding into the universal pointed dendrite* $(W, w)$.

*Proof.* Enlarge $X$ by attaching a segment at $P$ in order to assume that $P$ is a branch point of $X$. Theorem 6.2 yields an embedding $i\colon X \hookrightarrow W$. Then $i(P)$ is a branch point of $W$. By [Ch, Proposition 4.7], there is a homeomorphism $j\colon W \to W$ mapping $i(P)$ to $w$. Then $j \circ i$ is an embedding $(X, P) \to (W, w)$. □

**Proposition 6.5.** *Any dendrite admits a strong deformation retraction onto any of its points.*

*Proof.* In fact, a dendrite admits a strong deformation retraction onto any subcontinuum [Il]. □

## 7. Local dendrites

**7.1. Definition and basic properties.** A *local dendrite* is a continuum such that every point has a neighborhood that is a dendrite. Equivalently, a continuum is a local dendrite if and only if it is locally connected and contains at most a finite number of simple closed curves [Kur, §51, VII, Theorem 4(i)]. Local dendrites are generalizations of finite connected graphs, just as dendrites are generalizations of finite trees.

**Proposition 7.1.**

(a) *Every subcontinuum of a local dendrite is a local dendrite.*
(b) *An open subset of a local dendrite is arcwise connected if and only if it is connected.*
(c) *A connected open subset* $U$ *of a local dendrite is simply connected if and only if it contains no simple closed curve.*
(d) *A dendrite is the same thing as a simply connected local dendrite.*

*Proof.*

(a) This follows from the fact that every subcontinuum of a dendrite is a dendrite [Kur, §51, VI, Theorem 4].

(b) This follows from [Wh, II, (5.3)].

(c) If $U$ contains a simple closed curve $\gamma$, [BJ, Theorem on p. 174] shows that $\gamma$ cannot be deformed to a point, so $U$ is not simply connected. If $U$ does not contain a simple closed curve, then the image of any simple closed curve in $U$ is a dendrite, and hence by Proposition 6.5 is contractible, so $U$ is simply connected.

(d) This follows from (c). □

**7.2. Local dendrites and quasi-polyhedra.** Recall from [Bel, §4.1] that a connected locally compact Hausdorff space $X$ is called a (one-dimensional) *quasi-polyhedron* if all connected open subsets of $X$ are countable at infinity and $X$ admits a basis consisting of open subsets $U$ such that $\overline{U}-U$ is finite and such that, for every $x, y \in U$, there exists a unique closed subset contained in $U$ which is homeomorphic to the unit interval with endpoints $x$ and $y$. We now relate the notion of quasi-polyhedron to that of local dendrite.

**Proposition 7.2.**

(a) *A connected open subset of a local dendrite is a quasi-polyhedron.*

(b) *A compact metrizable quasi-polyhedron is the same thing as a local dendrite.*

(c) *A compact metrizable simply connected quasi-polyhedron is the same thing as a dendrite.*

(d) *A compact metrizable quasi-polyhedron is special in the sense of* [Bel, Definition 4.1.5].

*Proof.*

(a) Suppose that $V$ is a connected open subset of a local dendrite $X$. By [Kur, §51, VII, Theorem 1], each point $v$ of $V$ has arbitrarily small open neighborhoods $\mathcal{U}$ with finite boundary. We may assume that each $\mathcal{U}$ is contained in a dendrite. Since $V$ is locally connected, we may replace each $\mathcal{U}$ by its connected component containing $x$: this can only shrink its boundary. Now each $\mathcal{U}$, as a connected subset of a dendrite, is uniquely arcwise connected [Wh, p. 89, 1.3(ii)]. So these $\mathcal{U}$ satisfy [Bel, Definition 4.1.1(i)(a)]. By Proposition 7.8(a) (whose proof does not use anything from here on!), $X$ is homeomorphic to a compact subset of $\mathbb{R}^3$, so every open subset of $X$ is countable at infinity (i.e., a countable union of compact sets). Thus $V$ is a quasi-polyhedron.

(b) If $X$ is a local dendrite, it is a quasi-polyhedron by (a) and compact and metrizable by definition.
Conversely, suppose that $X$ is a compact metrizable quasi-polyhedron. In particular, $X$ is a continuum. Condition $(a_2)$ in [Be1, Definition 4.1.1] implies that $X$ is locally connected and covered by open subsets containing no simple closed curve. By compactness, this implies that there is a positive lower bound $\epsilon$ on the diameter of simple closed curves in $X$. By [Kur, §51, VII, Lemma 3], this implies that $X$ is a local dendrite.

(c) Combine (b) and Proposition 7.1(d).

(d) A dendrite is special since each partial ordering as in [Be1, Definition 4.1.5] arises from some $x \in X$, and we can take $\theta$ there to be a radial distance function as in [MO, Section 4.6], which applies since dendrites are locally arcwise connected and uniquely arcwise connected. A local dendrite is special since any simply connected sub-quasi-polyhedron is homeomorphic to a connected open subset of a dendrite. □

**7.3. The core skeleton.** By [Be1, Proposition 4.1.3(i)], any simply connected quasi-polyhedron $Q$ has a unique compactification $\widehat{Q}$ that is a simply connected quasi-polyhedron. The points of $\widehat{Q} - Q$ are called the *endpoints* of $Q$. Given a quasi-polyhedron $X$, Berkovich defines its *skeleton* $\Delta(X)$ as the complement in $X$ of the set of points having a simply connected quasi-polyhedral open neighborhood with a single endpoint [Be1, p. 76]. In the case of a local dendrite, we can characterize this subset in many ways: see Proposition 7.4.

**Lemma 7.3.** *Let $X$ be a local dendrite. Let $G$ be a subcontinuum of $X$ containing all the simple closed curves. Let $C$ be a connected component of $X - G$. Then $C$ is open in $X$ and is a simply connected quasi-polyhedron with one endpoint, and its closure $\overline{C}$ in $X$ is a dendrite intersecting $G$ in a single point.*

*Proof.* Since $X$ is locally connected, $X - G$ is locally connected, so $C$ is open. By Proposition 7.2(a), $C$ is a quasi-polyhedron. Since $C$ contains no simple closed curve, it is simply connected by Proposition 7.1(c).

The complement of $C \cup G$ is a union of connected components of $X - G$, so $C \cup G$ is closed, so it contains $\overline{C}$. Since $X$ is connected, $\overline{C} \neq C$, so $\#(\overline{C} \cap G) \geq 1$.

If $C$ had more than one endpoint, there would be an arc $\alpha$ in $\widehat{C}$ connecting two of them, passing through some $c \in C$ since $\widehat{C} - C$ is totally disconnected by [Be1, Proposition 4.1.3(i)]; the image of $\alpha$ under the induced map $\widehat{C} \to X$ together with an arc in $G$ connecting the images of the two endpoints would contain a simple closed curve passing through $c$, contradicting the hypothesis on $G$. Also, each point in $\overline{C} \cap G$ is the image of a point in $\widehat{C} - C$. Now $1 \leq \#(\overline{C} \cap G) \leq \#(\widehat{C} - C) \leq 1$, so equality holds everywhere. $\square$

**Proposition 7.4.** *Let $X$ be a local dendrite. Each of the following conditions defines the same closed subset $\Delta$ of $X$.*

(i) *If $X$ is a dendrite, $\Delta = \varnothing$; otherwise $\Delta$ is the smallest subcontinuum of $X$ containing all the simple closed curves.*

(ii) *The set $\Delta$ is the union of all arcs each endpoint of which belongs to a simple closed curve.*

(iii) *The set $\Delta$ is the skeleton $\Delta(X)$ defined in* [Bel, p. 76].

*Proof.* Let $L$ be the union of the simple closed curves in $X$. If $L = \varnothing$, then $X$ is a dendrite and (i), (ii), (iii) all define the empty set. So suppose that $L \neq \varnothing$.

For each pair of distinct components of $L$, there is at most one arc $\alpha$ in $X$ intersecting $L$ in two points, one from each component in the pair (otherwise there would be a simple closed curve not contained in $L$). Let $D$ be the union of all these arcs $\alpha$ with $L$. Any arc $\beta$ in $X$ with endpoints in $L$ must be contained in $D$, since a point of $\beta$ outside $D$ would be contained in some subarc $\beta'$ intersecting $L$ in just the endpoints of $\beta'$, which would then have to be some $\alpha$. Thus $D$ is the union of the arcs whose endpoints lie in $L$. By Proposition 7.1(b), $X$ is arcwise connected, so $D$ is arcwise connected. By definition, $D$ is a finite union of compact sets, so $D$ is a subcontinuum.

By Proposition 7.1(b), any subcontinuum $Y \subseteq X$ is arcwise connected, so if $Y$ contains $L$, then for each $\alpha$ as above, $Y$ contains an arc $\beta$ with the same endpoints as $\alpha$, and then $\beta = \alpha$ (otherwise there would be subarcs of $\alpha$ and $\beta$ whose union was a simple closed curve not contained in $L$); thus $Y \supseteq D$. Hence $D$ is the smallest subcontinuum containing $L$.

Let $\Delta$ be the $\Delta(X)$ of [Bel, p. 76]. If $x$ were a point in a simple closed curve $\gamma$ in $X$ with a neighborhood $Q$ as in the definition of $\Delta$, then $Q$ must contain $\gamma$, since otherwise $Q \cap \gamma$ would have a connected component homeomorphic to an open interval $I$, and the two points of $\widehat{I} - I$ would map to two distinct points of $\widehat{Q} - Q$, contradicting the choice of $Q$. Thus $\Delta \supseteq L$. But $D$ is the smallest subcontinuum containing $L$, so $\Delta \supseteq D$. On the other hand, Lemma 7.3 shows that the points of $X - D$ lie outside $\Delta$. Hence $\Delta = D$. $\square$

We call $\Delta$ the *core skeleton* of $X$, since in [HL, Section 10] the term "skeleton" is used more generally for any finite simplicial complex onto which $X$ admits a strong deformation retraction. If $\Delta \neq \varnothing$, then $\Delta$ is a finite connected graph with no vertices of degree less than or equal to 1 [Bel, Proposition 4.1.4(ii)].

### 7.4. $G$-dendrites.

**Proposition 7.5.** *For a subcontinuum $G$ of $X$, the following are equivalent.*

(i) *$G$ contains the core skeleton of $X$.*

(ii) *$G$ is a deformation retract of $X$.*

(iii) *$G$ is a strong deformation retract of $X$.*

(iv) *There is a retraction $r\colon X \to G$ such that there exists a homotopy $h\colon [0,1] \times X \to X$ between $h(0,x) = x$ and $h(1,x) = r(x)$ satisfying $r(h(t,x)) = r(x)$ for all $t$ and $x$ (i.e., "points are moved only along the fibers of $r$"); moreover, $r$ is unique, characterized by the condition that it maps each connected component $C$ of $X - G$ to the singleton $\overline{C} \cap G$.*

*Proof.* First we show that a retraction $r$ as in (iv) must be as characterized. Suppose that $C$ is a connected component of $X - G$. Any $c \in C$ is moved by the homotopy along a path ending on $G$, and if we shorten it to a path $\gamma$ so that it ends as soon as it reaches $G$ then $\gamma$ stays within $X - G$ until it reaches its final point $g$ and hence stays within $C$ until it reaches $g$; Hence $g \in \overline{C} \cap G$, and $r(c) = g$. Thus $r(C) \subseteq \overline{C} \cap G$. By Lemma 7.3, $\#(\overline{C} \cap G) = 1$, so $r$ is as characterized.

(i)$\Rightarrow$(iv): See [Bel, Proposition 4.1.6] and its proof.

(iv)$\Rightarrow$(iii): Trivial.

(iii)$\Rightarrow$(ii): Trivial.

(ii)$\Rightarrow$(i): The result of deforming the inclusion of a simple closed curve $\gamma$ in $X$ is a closed path whose image contains $\gamma$ [BJ, Theorem on p. 174], so if $G$ is a deformation retract of $X$, then $G$ must contain each simple closed curve, so $G$ contains the core skeleton. □

Given an embedding of local dendrites $G \hookrightarrow X$, call $X$ equipped with the embedding a *$G$-dendrite* if the image of $G$ satisfies the conditions of Proposition 7.5; we generally identify $G$ with its image. Let $\mathcal{D}_G$ be the category whose objects are $G$-dendrites and whose morphisms are embeddings extending the identity $1_G\colon G \to G$. Given a $G$-dendrite $X$ and $g \in G$, let $X_g$ be the fiber $r^{-1}(g)$ with the point $g$ distinguished; say that $g$ is a *sprouting point* if $X_g$ is not a point. Theorem 7.6 below makes precise the statement that any $G$-dendrite is obtained by attaching dendrites to countably many points of $G$. Recall that $\mathcal{P}$ is the category of pointed dendrites, defined in Section 6.3.

**Theorem 7.6.** *There is a fully faithful functor $F\colon \mathcal{D}_G \to \prod_{g\in G} \mathcal{P}$ sending a $G$-dendrite $X$ to the tuple of fibers $(X_g)_{g\in G}$, and its essential image consists of tuples $(D_g)$ such that $\{g \in G : \#D_g > 1\}$ is countable.*

*Proof.* Let $X$ be a $G$-dendrite. For each $g \in G$, the homotopy restricts to a contraction of $X_g$ to $g$, so $X_g$ is a (pointed) dendrite. By [Kur, §51, IV, Theorem 5 and §51, VII, Theorem 1], $\{g \in G : \#X_g > 1\}$ is countable.

The characterization of the retraction in Proposition 7.5(iv) shows that a morphism of $G$-dendrites $X \to Y$ respects the retractions, so it restricts to a morphism $X_g \to Y_g$ in $\mathcal{P}$ for each $g \in G$. This defines $F$.

Given $(D_g)_{g \in G} \in \prod_{g \in G} \mathcal{P}$ with $\{g \in G : \#D_g > 1\}$ countable, choose a metric $d_{D_g}$ on $D_g$ such that the diameters of the $D_g$ with $\#D_g > 1$ tend to 0 if there are infinitely many of them. Identify the distinguished point of $D_g$ with $g$. Let $X$ be the set $\coprod_{g \in G} D_g$ with the metric for which the distance between $x \in D_g$ and $x' \in D_{g'}$ is

$$\begin{cases} d_{D_g}(x, x'), & \text{if } g = g', \\ d_{D_g}(x, g) + d_G(g, g') + d_{D_{g'}}(g', x'), & \text{if } g \neq g'. \end{cases}$$

It is straightforward to check that $X$ is compact and locally connected and that the map $G \to X$ is an embedding. By Proposition 6.5, there is a strong deformation retraction of $D_g$ onto $\{g\}$; running these deformations in parallel yields a strong deformation retraction of $X$ onto $G$. Thus $X$ is a $G$-dendrite. Moreover, $F$ sends $X$ to $(D_g)_{g \in G}$. Thus the essential image is as claimed.

Given $X, Y \in \mathcal{D}_G$, and given morphisms $f_g \colon X_g \to Y_g$ in $\mathcal{P}$ for all $g \in G$, there exists a unique morphism $f \colon X \to Y$ in $\mathcal{D}_G$ mapped by $F$ to $(f_g)_{g \in G}$; namely, one checks that the union $f$ of the $f_g$ is a continuous injection, and hence an embedding. Thus $F$ is fully faithful. □

**7.5. The universal $G$-dendrite.** Let $G$ be a local dendrite. Given a countable subset $G_0 \subseteq G$, Theorem 7.6 yields a $G$-dendrite $W_{G,G_0}$ whose fiber at $g \in G$ is the universal pointed dendrite $(W, w)$ if $g \in G_0$ and a point if $g \notin G_0$. By Theorems 7.6 and 6.4, any $G$-dendrite with all sprouting points in $G_0$ admits a morphism to $W_{G,G_0}$.

Now let $G$ be a finite connected graph. Fix a countable dense subset $G_0 \subseteq G$ containing all vertices of $G$. Define $W_G := W_{G,G_0}$, and call it the *universal $G$-dendrite*. Its homeomorphism type is independent of the choice of $G_0$, since the possibilities for $G_0$ are permuted by the self-homeomorphisms of $G$ fixing its vertices. Any $G$-dendrite has its sprouting points contained in some $G_0$ as above (just take the union with a $G_0$ from above), so every $G$-dendrite embeds as a topological space into $W_G$.

**Theorem 7.7.** *Let $X$ be a local dendrite, and let $G$ be its core skeleton. Suppose that $G \neq \varnothing$, that the branch points of $X$ are dense in $X$, and that there are $\aleph_0$ branches at each branch point. Then $X$ is homeomorphic to $W_G$.*

*Proof.* The vertices of $G$ of degree 3 or more are among the branch points of $X$. After applying a homeomorphism of $G$ (to shift degree 2 vertices), we may assume that *all* the vertices of $G$ are branch points of $X$. Since the branch points of $X$ are dense in $X$, the sprouting points must be dense in $G$. For each sprouting point $g \in G$, the fiber $X_g$ satisfies the hypotheses of Theorem 6.1, so $X_g$ is the universal pointed dendrite. Thus $X$ is homeomorphic to $W_G$, by construction of the latter. □

### 7.6. Euclidean embeddings.

**Proposition 7.8.**

(a) *Every local dendrite embeds in* $\mathbb{R}^3$.

(b) *Let* $X$ *be a local dendrite, and let* $G \subseteq X$ *be a finite connected graph containing all the simple closed curves. Then the following are equivalent:*

  (i) $X$ *embeds into* $\mathbb{R}^2$.

  (ii) $G$ *embeds into* $\mathbb{R}^2$.

  (iii) $G$ *does not contain a subgraph isomorphic to a subdivision of the complete graph* $K_5$ *or the complete bipartite graph* $K_{3,3}$.

*Proof.*

(a) A local dendrite is a regular continuum [Kur, §51, VII, Theorem 1], and hence of dimension 1, so it embeds in $\mathbb{R}^3$ as discussed in Remark 3.2.

(b) See [Ku]. □

## 8. Berkovich curves

Finally, we build on [Be1] (especially Section 4 therein) and the theory of local dendrites to describe the homeomorphism type of a Berkovich curve. See also the forthcoming book by Ducros [Du], which will contain a systematic study of Berkovich curves.

**Theorem 8.1.** *Let* $K$ *be a complete valued field having a countable dense subset. Let* $V$ *be a projective* $K$*-scheme of pure dimension* 1.

(a) *The topological space* $V^{\mathrm{an}}$ *is a finite disjoint union of local dendrites.*

(b) *Suppose that* $V$ *is also smooth and connected, and that* $K$ *has nontrivial value group.*

  (i) *If* $V^{\mathrm{an}}$ *is simply connected, then* $V^{\mathrm{an}}$ *is homeomorphic to the Ważewski universal dendrite* $W$.

  (ii) *If* $V^{\mathrm{an}}$ *is not simply connected, let* $G$ *be its core skeleton; then* $V^{\mathrm{an}}$ *is homeomorphic to the universal* $G$*-dendrite* $W_G$.

*Proof.*

(a) We may assume that $V$ is connected, so $V^{\mathrm{an}}$ is connected by [Bel, Theorem 3.4.8(iii)]. Also, $V^{\mathrm{an}}$ is compact by [Bel, Theorem 3.4.8(ii)], It is metrizable by Remark 1.5 (or Theorem 1.1). It is a quasi-polyhedron by [Bel, Theorem 4.3.2 and the proof of Corollary 4.3.3]: indeed, one may assume that $K$ is algebraically closed and $V$ is reduced; since $V$ is obtained from its normalization by glueing together a finite number of closed points, we may assume that $V$ is smooth; this case follows directly from [Bel, Theorem 4.3.2]. So $V^{\mathrm{an}}$ is a local dendrite by Proposition 7.2.

(b) Let $k$ be the residue field of $K$. Since $K$ has a countable dense subset, $k$ is countable, so any $k$-curve has exactly $\aleph_0$ closed points.
First suppose that $K$ is algebraically closed. In particular $K$ has dense value group. Choose a semistable decomposition of $V^{\mathrm{an}}$ (see [BPR, Definition 5.15]). Each open ball and open annulus in the decomposition is homeomorphic to an open subspace of $(\mathbb{P}^1_K)^{\mathrm{an}}$, in which the branch points (type (2) points in the terminology of [Bel, 1.4.4]) are dense by the assumption on the value group, so the branch points are dense in $V^{\mathrm{an}}$. At each branch point, the branches are in bijection with the closed points of a $k$-curve by [BPR, Lemma 5.66(3)], so their number is $\aleph_0$.
Now suppose that $K$ is not necessarily algebraically closed. Let $K'$ be the completion of an algebraic closure of $K$. Then [Bel, Corollary 1.3.6] implies that $V^{\mathrm{an}}$ is the quotient of $(V_{K'})^{\mathrm{an}}$ by the absolute Galois group of $K$. It follows that the branch points of $V^{\mathrm{an}}$ are the images of the branch points of $(V_{K'})^{\mathrm{an}}$, and that the branches at each branch point of $V^{\mathrm{an}}$ are in bijection with the closed points of some curve over a finite extension of $k$. Thus, as for $(V_{K'})^{\mathrm{an}}$, the branch points of $V^{\mathrm{an}}$ are dense, and there are $\aleph_0$ branches at each branch point.
Finally, according to whether $G$ is simply connected or not, Theorem 6.1 or Theorem 7.7 shows that $V^{\mathrm{an}}$ has the stated homeomorphism type. □

**Corollary 8.2.** *Let $K$ be a complete valued field having a countable dense subset and dense value group. Then $(\mathbb{P}^1_K)^{\mathrm{an}}$ is homeomorphic to $W$.*

*Proof.* It is simply connected by [Bel, Theorem 4.2.1], so Theorem 8.1(b)(i) applies. □

**Remark 8.3.** Any finite connected graph with no vertices of degree less than or equal to 1 can arise as the core skeleton $G$ in Theorem 8.1(b)(ii): see [Bel, proof of Corollary 4.3.4]. In particular, there exist smooth projective curves $V$ such that $V^{\mathrm{an}}$ cannot be embedded in $\mathbb{R}^2$.

**Remark 8.4.** Theorem 8.1 also lets us understand the topology of Berkovich spaces associated to curves that are only *quasi-projective*. Let $U$ be a quasi-projective curve. Write $U = V - Z$ for some projective curve $V$ and finite subscheme $Z \subseteq V$. Then $Z^{\mathrm{an}}$ is a closed subset of $V^{\mathrm{an}}$ with one point for each closed point of $Z$, and $U^{\mathrm{an}} = V^{\mathrm{an}} - Z^{\mathrm{an}}$.

**Remark 8.5.** The smoothness assumption in Theorem 8.1(b) can be weakened to the statement that the normalization morphism $\widetilde{V} \to V$ has no fibers with three or more schematic points.

**Remark 8.6.** If in Theorem 8.1(b) we drop any of the hypotheses, then the result fails; we describe the situations that arise.

- If $V$ is the non-smooth curve consisting of three copies of $\mathbb{P}^1_K$ attached at a $K$-point of each, then $V^{\mathrm{an}}$ consists of three copies of $W$ attached in the same way; this is a dendrite, but it has a branch point of order $3$, so it cannot be homeomorphic to $W$. More generally, if the normalization $\widetilde{V}$ has three distinct schematic points above some point $a$ of $V$, the same argument applies.
- If $V$ is disconnected, then so is $V^{\mathrm{an}}$, so it cannot be homeomorphic to $W$ or $W_G$. In this case, $V^{\mathrm{an}}$ is the disjoint union of the analytifications of the connected components of $V$.
- Suppose that $V$ is smooth and connected, but $K$ has trivial value group. Then $V^{\mathrm{an}}$ is a dendrite consisting of $\aleph_0$ intervals emanating from one branch point; cf. [Be2, p. 71]. Equivalently, $V^{\mathrm{an}}$ is the one-point compactification of $|V| \times [0,\infty)$, where $|V|$ is the set of closed points of $V$ with the discrete topology.

**Remark 8.7.** As is well-known to experts [Th, BPR], there is a metrized variant of Theorem 8.1. We recall a few definitions; cf. [MNO]. An $\mathbb{R}$*-tree* is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of $\mathbb{R}$. Let $A$ be a countable subgroup of $\mathbb{R}$, and let $A_{\geq 0}$ (resp. $A_{>0}$) be the set of nonnegative (resp. positive) numbers in $A$. An $A$*-tree* is an $\mathbb{R}$-tree $X$ equipped with a point $x \in X$ such that the distance from each branch point to $x$ lies in $A$.

More generally, we may introduce variants that are not simply connected. Let us define an $\mathbb{R}$*-graph* to be an arcwise connected metric space $X$ such that each arc of $X$ is isometric to a subarc of $\mathbb{R}$ and $X$ contains at most finitely many simple closed curves. Define an $A$*-graph* to be an $\mathbb{R}$-graph $X$ equipped with a point $x \in X$ such that the length of every arc from $x$ to a branch point or to itself is in $A$. Given an $A$-graph $(X,x)$, let $B(X)$ be the set of points

$y \in B$ not of degree 1 such that $y$ is an endpoint of an arc of length in $A_{\geq 0}$ emanating from $x$. Then let $\mathcal{E}(X)$ be the $A$-graph obtained by attaching $\aleph_0$ isometric copies of $[0, \infty)$ and of $[0, a]$ for each $a \in A_{>0}$ to each $y \in B(X)$ (i.e., identify each 0 with $y$). Let $\mathcal{E}^n(X) := \mathcal{E}(\mathcal{E}(\cdots(\mathcal{E}(X))\cdots))$. The direct limit of the $\mathcal{E}^n(X)$ is an $A$-graph $\mathcal{W}_X^A$. If $X$ is a point, define $\mathcal{W}^A := \mathcal{W}_X^A$, which is a universal separable $A$-tree in the sense of [MNO, Section 2], because it contains the space obtained by attaching only copies of $[0, \infty)$ at each stage; the latter is the universal separable $A$-tree constructed in [MNO, Theorem 2.6.1].

Let $K$ be a complete algebraically closed valued field having a countable dense subset. Let $A$ be the value group of $K$, expressed as a $\mathbb{Q}$-subspace of $\mathbb{R}$. Let $V$ be a projective $K$-scheme of pure dimension 1. Let $V^{\text{an}-}$ be the subset of $V^{\text{an}}$ consisting of the complement of the type (1) points (the points corresponding to closed points of $V$). Then $V^{\text{an}-}$ admits a canonical metric, whose existence is related to the fact that on the segments of the skeleta of $V^{\text{an}}$, away from the endpoints, one has an integral affine structure [KS, Section 2]. If $V^{\text{an}-}$ is simply connected, then $V^{\text{an}-}$ is isometric to $\mathcal{W}^A$; otherwise $V^{\text{an}-}$ is isometric to $\mathcal{W}_G^A$, where $G$ is the core skeleton of $V^{\text{an}}$ with the induced metric.

**Warning 8.8.** The metric topology on $V^{\text{an}-}$ is strictly stronger than the subspace topology on $V^{\text{an}-}$ induced from $V^{\text{an}}$: see [FJ, Chapter 5] and [BR, Section B.6]. Nevertheless, when $V$ is smooth and complete, the topological space $V^{\text{an}}$ can be recovered from the metric space $V^{\text{an}-}$.

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