6.3 BOUNDED COHOMOLOGY AND THE MILNOR-WOOD INEQUALITY

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **03.05.2024**

$eu(\phi)$ in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each $1 \leq i \leq g$, choose lifts $\widetilde{a}_i$ and $\widetilde{b}_i$ of $\phi(a_i)$ and $\phi(b_i)$. Now compute the product of commutators $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\dots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number $eu(\phi)$.

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus $g > 1$ can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space $\mathcal{H}$ can be equipped with a metric of curvature -1 whose group of orientation preserving isometries is precisely $\mathrm{PSL}(2,\mathbf{R})$. Moreover, any complete simply connected riemannian surface of curvature -1 is isometric to $\mathcal{H}$. Hence there are embeddings ϕ of the fundamental group Γ_g of a closed oriented surface of genus $g > 1$ in $\mathrm{PSL}(2,\mathbf{R})$ such that the corresponding action of Γ_g on $\mathcal{H}$ is free, proper and cocompact. Since we know that $\mathrm{PSL}(2,\mathbf{R})$ is a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can compute the corresponding Euler number $eu(\phi)$. The result of the computation is $2g-2$. Note that each element of $\phi(\Gamma_g)$ is hyperbolic since the action is free and cocompact so that the rotation number of every element of $\phi(\Gamma_g)$ is 0. So we are in a situation in which the topological invariant $eu(\phi)$ is not 0 but the rotation number invariants are trivial; a situation different from the case where $\Gamma = \mathbf{Z}$.

6.3 Bounded cohomology and the Milnor-Wood inequality

It was observed very early that the Euler class of a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ cannot be arbitrary. Milnor and Wood proved the following [51, 71].

THEOREM 6.1 (Milnor-Wood). *Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$ and $\phi\colon \Gamma_g \to \mathrm{Homeo}_+(\mathbf{S}^1)$ be any homomorphism. Then the Euler number satisfies $|eu(\phi)| \leq 2g-2$.*

Proof. We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that $eu(\phi)$ is the translation number of the homeomorphism $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\dots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. We also know that the translation number function τ is a quasi-homomorphism, *i.e.* there is some inequality of the form $|\tau(\widetilde{f}_1\widetilde{f}_2)-\tau(\widetilde{f}_1)-\tau(\widetilde{f}_2)| \leq D$ for some D. We also know that $\tau(\widetilde{f}^{-1}) = -\tau(\widetilde{f})$. So, if we evaluate τ on this element, we get a bound of the form $|eu(\phi)| \leq (4g-1)D$. This is not quite the bound given in the theorem but this explains the idea of the proof: to get the exact bound, one should be a little bit more clever! □

In [17], Eisenbud, Hirsch and Neumann gave a much more precise result that we would like to mention here. If $\widetilde{f}$ is an element of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, define $\underline{m}(\widetilde{f}) = \inf(\widetilde{f}(x)-x)$ and $\overline{m}(\widetilde{f}) = \sup(\widetilde{f}(x)-x)$. Note that $\underline{m}(\widetilde{f}) \leq \tau(\widetilde{f}) \leq \overline{m}(\widetilde{f})$ and $0 \leq \overline{m}(\widetilde{f}) - \underline{m}(\widetilde{f}) < 1$.

THEOREM 6.2 (Eisenbud, Hirsch, Neumann). *An element $\widetilde{f}$ of the group $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ can be written as a product of $g \geq 1$ commutators if and only if $\underline{m}(\widetilde{f}) < 2g-1$ and $1-2g < \overline{m}(\widetilde{f})$.*

Any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ has at least one lift $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $-1 < \underline{m}(\widetilde{f}) \leq \overline{m}(\widetilde{f}) < 1$ so that it can be written as one commutator. It follows that every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be written as a commutator. We mentioned this fact earlier.

In [25], we put these inequalities in the context of bounded cohomology, which was introduced by Gromov (see [30] for many geometrical motivations). Consider again an abstract group Γ and let $A = \mathbf{Z}$ or $\mathbf{R}$. Then define a *bounded* k-cochain as a bounded homogeneous map from Γ^{k+1} to A. This defines a sub A-module of $C^k(\Gamma, A)$ denoted by $C_b^k(\Gamma, A)$. It is clear that the coboundary d_k of a bounded k-cochain is a bounded $(k+1)$-cochain so that we can define the cohomology of this new differential complex, that is called the *bounded cohomology of* Γ *with coefficients in* A and denoted by $H_b^k(\Gamma, A)$. We have obvious maps from $H_b^k(\Gamma, A)$ to $H^k(\Gamma, A)$ obtained by "forgetting" that a cocycle is bounded. In general these maps are neither injective nor surjective. See [35, 36] for a detailed algebraic background on this cohomology.

The degree 1 case is trivial. A cocycle is given by a bounded homomorphism from Γ to A and is therefore trivial. Hence $H_b^1(\Gamma, A) = 0$ for any group Γ.

The degree 2 case is the most interesting for us. Let us look first at $H_b^2(\mathbf{Z}, \mathbf{R})$. Consider a bounded 2-cocycle c on $\mathbf{Z}$ with values in $\mathbf{R}$. Since we know that $H^2(\mathbf{Z}, \mathbf{R}) = 0$, we know that c is the coboundary of a 1-cochain of the form $u(n_1, n_2) = \overline{u}(n_1 - n_2)$ for some function $\overline{u} \colon \mathbf{Z} \to \mathbf{R}$. The fact that c is bounded means precisely that $\overline{u}$ is a quasi-homomorphism from $\mathbf{Z}$ to $\mathbf{R}$. We know that this implies the existence of a real number τ such that $\overline{u}(n) - n\tau$ is bounded. Now, if we define $\overline{v}(n) = \overline{u}(n) - n\tau$, then the coboundary of the *bounded* 1-cochain $v(n_1,n_2)=\overline{v}(n_1-n_2)$ is c. We have shown that $H_b^2(\mathbf{Z},\mathbf{R})=0$.

For a general group Γ, let us define $QM(\Gamma)$ as being the vector space of quasi-homomorphisms from Γ to $\mathbf{R}$. Say that a quasi-homomorphism is trivial if it differs from some homomorphism by a bounded amount. It follows from the definitions and the previous argument that the kernel of the map from $H_b^2(\Gamma,\mathbf{R})$ to $H^2(\Gamma,\mathbf{R})$ is precisely the quotient of $QM(\Gamma)$ by the subspace of trivial quasi-homomorphisms. This gives some intuition about the group $H_b^2(\Gamma,\mathbf{R})$.

Let us compute now some examples with coefficients in $\mathbf{Z}$. Start with $H_b^2(\mathbf{Z},\mathbf{Z})$. Let c be a bounded integral 2-cocycle. We know that it is the coboundary of a 1-cochain of the form $u(n_1,n_2)=\overline{u}(n_1-n_2)$ for some function $\overline{u}\colon \mathbf{Z}\to\mathbf{Z}$. Again, we know that there is a real number τ such that $\overline{u}(n)-n\tau$ is bounded but if we define $\overline{v}(n)=\overline{u}(n)-n\tau$ the 1-cochain v is not integral unless τ is an integer! For each real number τ, define c_τ to be the coboundary of the integral 1-cochain $v_\tau(n_1,n_2)=[(n_1-n_2)\tau]$ where $[\,]$ denotes the integral part of a real number. It is clear that c_τ is bounded (by 1) and our previous computations show that every bounded integral 2-cocycle in $\mathbf{Z}$ is cohomologous to some c_τ for some τ. Moreover, it is clear that c_{τ_1} and c_{τ_2} define the same element in $H_b^2(\mathbf{Z},\mathbf{Z})$ if and only if $\tau_1-\tau_2$ is an integer. Summing up, we showed that $H_b^2(\mathbf{Z},\mathbf{Z})$ is isomorphic to $\mathbf{R}/\mathbf{Z}$. We hope that the reader will recognize that the rotation number is showing up...

As a matter of fact, the argument that we presented is more general and shows immediately that for any group Γ, the kernel of the map from $H_b^2(\Gamma,\mathbf{Z})$ to $H_b^2(\Gamma,\mathbf{R})$ is precisely the quotient $H^1(\Gamma,\mathbf{R})/H^1(\Gamma,\mathbf{Z})$. (Recall that $H^1(\Gamma,A)$ is the set of homomorphisms from Γ to A.)

We now come to the construction of an invariant of a group action on the circle that combines the rotation numbers and the Euler class. Let us look again at the central extension

$$0\longrightarrow \mathbf{Z}\longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)\longrightarrow \mathrm{Homeo}_+(\mathbf{S}^1)\longrightarrow 1$$

and let us try to find some 2-cocycle representing its Euler class (see also [38]). We know that we should choose a set theoretical section s to p. It turns out that there is a natural choice of such a section. Indeed, let $f\in\mathrm{Homeo}_+(\mathbf{S}^1)$, then among the elements in $p^{-1}(f)\in\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, there is only one, denoted by $\sigma(f)$, which is such that $\sigma(f)(0)$ lies in the interval $[0,1[\subset\mathbf{R}$. This σ will be our preferred section. Let us try to evaluate the associated 2-cocycle c on $\mathrm{Homeo}_+(\mathbf{S}^1)$. By definition the associated inhomogeneous cocycle $\overline{c}$ is:

$$\overline{c}(f_1,f_2)=\sigma(f_1f_2)^{-1}\sigma(f_1)\sigma(f_2).$$

The main (easy) observation is that the cocycle c is bounded. More precisely:

LEMMA 6.3. *The 2-cocycle c takes only the two values 0 and 1.*

Proof. By definition $\sigma(f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1)(\sigma(f_2)(0))$ is in the interval $[\sigma(f_1)(0), \sigma(f_1)(0)+1[$ which is contained in $[0,2[$. We know that $\sigma(f_1 f_2)$ and $\sigma(f_1)\sigma(f_2)$ are lifts of the same element $f_1 f_2$ and that $\sigma(f_1 f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1 f_2)^{-1}\sigma(f_1)\sigma(f_2)$ is the translation by 0 or 1. □

Hence, for this choice of section σ, the associated 2-cocycle c is bounded and integral. Thus, we have defined an element of $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ that we call the *bounded Euler class*. It may seem that the definition depends on the choice of the origin 0 on the line but the reader will easily check that a modification of the origin would change the section σ by a bounded amount so that the bounded integral cohomology class is indeed well defined. If we have a homomorphism ϕ from a group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ we can pull back this bounded Euler class. We get an element in $H_b^2(\Gamma, \mathbf{Z})$ that we still denote by $eu(\phi)$ and that we call the bounded Euler class of the homomorphism ϕ. In case $\Gamma = \mathbf{Z}$, it should now be clear that the corresponding bounded Euler class in $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is exactly the rotation number of the homeomorphism $\phi(1)$. Hence we have proved the following:

THEOREM 6.4 ([25]). *There is a class eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ such that:*

1) *For every homomorphism $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ the image of $\phi^*(eu) \in H_b^2(\Gamma, \mathbf{Z})$ in $H^2(\Gamma, \mathbf{Z})$ under the canonical map is the Euler class.*

2) *If $\phi \colon \mathbf{Z} \to \mathrm{Homeo}_+(\mathbf{S}^1)$ is a homomorphism then $\phi^*(eu) \in H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is the rotation number of $\phi(1)$.*

3) *$\phi^*(eu)$ is a topological invariant, i.e. if ϕ_1 and ϕ_2 are two homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ which are conjugate by an orientation preserving homeomorphism, then $\phi_1^*(eu) = \phi_2^*(eu)$ in $H_b^2(\Gamma, \mathbf{Z})$.*

In other words, the bounded Euler class is a topological invariant which combines the Euler class and the rotation number.

We now show that this new invariant for a group action is as powerful as the rotation number was for a single homeomorphism. Let us begin by the most interesting case.

THEOREM 6.5 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense on the circle. Assume that the bounded Euler classes are equal: $\phi_1^*(eu) = \phi_2^*(eu)$. Then ϕ_1 and ϕ_2 are conjugate by an orientation preserving homeomorphism.*

Proof. This is very similar to the corresponding statement for rotation numbers: compare with the proof of 5.9. Since $\phi_1^*(eu) = \phi_2^*(eu)$ then in particular the Euler classes in $H^2(\Gamma, \mathbf{Z})$ are equal, which means that ϕ_1 and ϕ_2 define the same central extension $\widetilde{\Gamma}$. In other words, there is a central extension $0 \to \mathbf{Z} \to \widetilde{\Gamma} \to \Gamma \to 1$ and homomorphisms $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ from $\widetilde{\Gamma}$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\widetilde{\phi}_1$ and $\widetilde{\phi}_1$ map the generator 1 of $\mathbf{Z}$ on the translation by 1 and such that the induced homomorphisms from $\widetilde{\Gamma}/\mathbf{Z} \simeq \Gamma$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)/\mathbf{Z} \simeq \mathrm{Homeo}_+(\mathbf{S}^1)$ are ϕ_1 and ϕ_2. The assumption that the bounded classes agree means in fact that we can choose those homomorphisms in such a way that for each x in $\mathbf{R}$, the points $\widetilde{\phi}_1(\widetilde{\gamma})\widetilde{\phi}_2(\widetilde{\gamma})^{-1}(x)$ are bounded independently of $\widetilde{\gamma}$ in $\widetilde{\Gamma}$. We now define $\widetilde{h}(x)$ to be the upper bound of this bounded set. This map $\widetilde{h}$ is increasing, commutes with integral translations, and conjugates $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$. The jump and plateau sets of $\widetilde{h}$ are open sets invariant under $\widetilde{\phi}_1(\widetilde{\Gamma})$ and $\widetilde{\phi}_2(\widetilde{\Gamma})$ respectively. By our assumption these open sets are empty so that $\widetilde{h}$ is a homeomorphism which induces a conjugacy between ϕ_1 and ϕ_2. For more details, see [25]. $\square$

In case the group $\phi(\Gamma)$ does not have all its orbits dense, we saw in 5.6 that there are two possibilities: $\phi(\Gamma)$ can have a finite orbit or $\phi(\Gamma)$ can have an exceptional minimal set. In the second case, we also saw that there is a canonical way of "collapsing" the connected components of the complement of the exceptional minimal set to construct another homomorphism $\overline{\phi}$ which has all its orbits dense: this is the associated "minimal" homomorphism (see 5.8).

Suppose now that $\phi(\Gamma)$ has a finite orbit consisting of k elements. Then, every element of $\phi(\Gamma)$ must permute these k points cyclically so that we get a homomorphism $r\colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$. It is clear that two finite orbits of $\phi(\Gamma)$ have the same number of points and define the same r: we call this r the *cyclic structure of the finite orbits.* Conversely, consider a homomorphism $r\colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$ and the corresponding action on the circle by rotations of order k. The bounded Euler class of this action is an element of $H_b^2(\Gamma, \mathbf{Z})$: we call these elements the *rational elements* in $H_b^2(\Gamma, \mathbf{Z})$. It is not difficult to see that an element in $H_b^2(\Gamma, \mathbf{Z})$ is rational if and only if its pull-back on some finite index subgroup is trivial.

Now, we can state the general result which is the exact analogue of what has been done in 5.9 for the rotation number. We don't give the proof: it can be found in [25] (in a slightly different terminology and with small mistakes...), but the reader should now be in a condition to fill in the missing details by himself.

THEOREM 6.6 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Assume that the bounded Euler classes $\phi_1^*(eu) = \phi_2^*(eu)$ are equal to the same class c in $H_b^2(\Gamma, \mathbf{Z})$.*

1) *If c is a rational class, then $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits with the same cyclic structure.*

2) *If c is not rational, then the associated minimal homomorphisms $\overline{\phi}_1$ and $\overline{\phi}_2$ are conjugate.*

Conversely, if $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits of the same cyclic structure or if they have no finite orbit and their associated minimal homomorphisms are conjugate (by an orientation preserving homeomorphism), then they have the same bounded Euler class.

Note in particular that the bounded Euler class of an action vanishes if and only if there is a point on the circle which is fixed by all the elements of the group.

6.4 EXPLICIT BOUNDS ON THE EULER CLASS

Since we know that the bounded Euler class of an action contains almost all the topological information, it is very natural to try to determine the part of $H_b^2(\Gamma, \mathbf{Z})$ which corresponds to the bounded Euler classes of all actions of Γ on the circle. In the case $\Gamma = \mathbf{Z}$, we know that $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ and that every class corresponds to an action (by rotations). However, in the case where Γ is the fundamental group of a closed oriented surface of genus $g \geq 1$, the Milnor-Wood inequality shows that even the usual Euler class in $H^2(\Gamma, \mathbf{Z}) = \mathbf{Z}$ has to satisfy some inequality.

Given a bounded cochain c in $C_b^k(\Gamma, \mathbf{R})$, we define its norm $\|c\|$ as the supremum of the absolute value of $c(\gamma_0, \ldots, \gamma_k)$. Then we define the "norm" of a bounded cohomology class with real coefficients as the infimum of the norms of cocycles that represent it. We should be aware of the fact that this norm is not really a norm but is merely a semi-norm: a non zero class might