

## §2. Notations and some known facts

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(Thm. 1) by a very simple general argument that the operator  $\mathcal{F}_p$  in (3) is surjective only in the following obvious cases: (i)  $p = p' = 2$  or (ii)  $G$  finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group  $G$  whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary  $L^p$ -spaces (stated in § 2). From this we deduce fairly simply that for any infinite locally compact commutative group  $G$ , the inequality (1) cannot be extended to the case  $2 < p < \infty$ ; the exact statement is given as Thm. 2 in § 3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in § 2, state and prove the facts alluded to above in § 3 and add a few historical comments in § 4; a short appendix (§ 5) is added to explain the  $L^p$ -isomorphism theorem stated in § 2.

We have not tried to extend our theorems to the case of  $G$  non-commutative, using for  $\widehat{G}$  the set of all equivalence classes of continuous unitary irreducible representations of  $G$ . For  $G$  compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in § 4.

## §2. NOTATIONS AND SOME KNOWN FACTS

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple  $(X, \Sigma, \mu)$  where  $\Sigma$  is a  $\sigma$ -algebra of subsets of the abstract set  $X$  and  $\mu: \Sigma \rightarrow [0, \infty]$  is a  $\sigma$ -additive positive measure; no finiteness or  $\sigma$ -finiteness conditions will be imposed a priori on  $\mu$ . Then  $L^p(\mu)$ ,  $1 \leq p \leq \infty$ , will denote the usual Banach space associated with  $\Sigma$ -measurable complex-valued functions  $f$  defined on  $X$  with  $\|f\|_p < \infty$ ,  $\|f\|_p$  being the standard  $L^p$ -norm with respect to  $\mu$ . If  $G$  is a locally compact commutative group (always supposed to be Hausdorff),  $L^p(G)$ ,  $1 \leq p \leq \infty$ , will stand for the associated  $L^p$ -space obtained by fixing some Haar (invariant) measure on  $G$ , and  $\widehat{G}$  will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma: G \rightarrow \mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}.$$

For a given Haar measure on  $G$ , the Haar measure on  $\widehat{G}$  will always be fixed in such a way that the Plancherel formula be valid in  $L^2(G)$ ; if  $f \in L^1(G)$ ,  $\widehat{f}$  will be defined by (2) above.

Recall that for any measure space  $(X, \Sigma, \mu)$ , the dual Banach space of  $L^p(\mu)$  is  $L^{p'}(\mu)$  whenever  $1 < p < \infty$ , i.e. in symbols

$$(4) \quad (L^p(\mu))' = L^{p'}(\mu)$$

whether  $\mu$  is  $\sigma$ -finite or not. Here and elsewhere,

$$p' = \frac{p}{p-1}, \quad 1 < p < \infty,$$

and  $1' = \infty$ ; for (4) to hold for  $p = 1$ ,  $p' = \infty$ , one needs some conditions on  $\mu$  ( $\sigma$ -finiteness of  $\mu$  is sufficient but not necessary). Nevertheless,

$$(L^p(G))' = L^{p'}(G)$$

holds for all  $p, 1 \leq p < \infty$ , and any locally compact group  $G$ . We shall not, however, need this fact.

Two Banach spaces  $E, F$  are called isomorphic if there is an isomorphism  $u: E \rightarrow F$  where  $u$  is a continuous linear bijection; it is well-known that  $u^{-1}: F \rightarrow E$  is then automatically continuous. The following is proved in [C]: *if  $(X, \Sigma, \mu)$ ,  $(Y, \mathcal{J}, \nu)$  are any two measure spaces and  $1 \leq p, q \leq \infty$  then  $L^p(\mu)$  isomorphic to  $L^q(\nu)$  implies necessarily that  $p = q$  provided that  $L^p(\mu)$  or  $L^q(\nu)$  is infinite dimensional.*

We shall refer to this statement as the  *$L^p$ -isomorphism theorem*; as indicated in §5, this is an easy consequence of the theory of types and cotypes for Banach spaces. The same reasoning proves (cf. §5) that if  $(X, \Sigma, \mu)$  is any measure space such that  $L^1(\mu)$  is infinite dimensional and  $Y$  is any locally compact Hausdorff space then  $L^1(\mu)$  cannot be isomorphic to  $C_0(Y)$  (or to  $C(Y)$ ). Here  $C(Y)$  is the Banach space of all bounded complex-valued functions on  $Y$  endowed with the sup norm, and  $C_0(Y)$  is the subspace of those continuous complex-valued functions in  $Y$  which vanish at  $\infty$ ; if  $Y$  is compact, we put  $C_0(Y) = C(Y)$ .

Recall that if  $f \in L^1(G)$ ,  $G$  any locally compact commutative group, then  $\widehat{f} \in C_0(\widehat{G})$ ; cf. [HR] vol. 2, p. 212; this fact is sometimes referred to as the Riemann-Lebesgue lemma for  $G$ .