

# §1. Introduction

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## REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY

by Srishti D. CHATTERJI

### § 1. INTRODUCTION

A standard version of the Hausdorff-Young inequality for a locally compact commutative group  $G$  can be given as follows: for a fixed Haar measure in  $G$ , let  $f \in L^1(G) \cap L^2(G)$ ; if  $1 \leq p \leq 2$ ,  $p' = p/(p - 1)$ , then

$$(1) \quad \|\widehat{f}\|_{p'} \leq \|f\|_p$$

where

$$(2) \quad \widehat{f}(\gamma) = \int_G f(x) \overline{\gamma(x)} dx, \quad \gamma \in \widehat{G},$$

$\widehat{G}$  being the dual group of  $G$ , endowed with a Haar measure which is such that for  $p = p' = 2$ , there is equality in (1); that this last condition can be met is one form of Plancherel's theorem in  $L^2(G)$ . Note that, for  $1 \leq p \leq 2$ ,  $\|f\|_p < \infty$  if  $f$  is in  $L^1(G) \cap L^2(G)$ , the latter space being dense in each  $L^p(G)$ ,  $1 \leq p \leq 2$ . Hence, because of the Hausdorff-Young inequality (1), the Fourier transform  $\mathcal{F}_p f$  can be defined uniquely for all  $f \in L^p(G)$ ,  $1 \leq p \leq 2$ , in such a way that

$$(3) \quad \mathcal{F}_p: L^p(G) \rightarrow L^{p'}(\widehat{G})$$

is a linear contraction with  $\mathcal{F}_p f = \widehat{f}$  for all  $f$  in  $L^1(G) \cap L^2(G)$ . It is known that, for each  $p \in [1, 2]$ ,  $\mathcal{F}_p$  is injective and that if  $f \in L^{p_1}(G) \cap L^{p_2}(G)$ ,  $1 \leq p_1, p_2 \leq 2$ , then  $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f$  a.e. on  $\widehat{G}$ ; see [HR] vol. 2, chap. VIII ((31.26), p. 229; (31.31), p. 231). The purpose of the present note is to prove

(Thm. 1) by a very simple general argument that the operator  $\mathcal{F}_p$  in (3) is surjective only in the following obvious cases: (i)  $p = p' = 2$  or (ii)  $G$  finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group  $G$  whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary  $L^p$ -spaces (stated in § 2). From this we deduce fairly simply that for any infinite locally compact commutative group  $G$ , the inequality (1) cannot be extended to the case  $2 < p < \infty$ ; the exact statement is given as Thm. 2 in § 3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in § 2, state and prove the facts alluded to above in § 3 and add a few historical comments in § 4; a short appendix (§ 5) is added to explain the  $L^p$ -isomorphism theorem stated in § 2.

We have not tried to extend our theorems to the case of  $G$  non-commutative, using for  $\widehat{G}$  the set of all equivalence classes of continuous unitary irreducible representations of  $G$ . For  $G$  compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in § 4.

## §2. NOTATIONS AND SOME KNOWN FACTS

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple  $(X, \Sigma, \mu)$  where  $\Sigma$  is a  $\sigma$ -algebra of subsets of the abstract set  $X$  and  $\mu: \Sigma \rightarrow [0, \infty]$  is a  $\sigma$ -additive positive measure; no finiteness or  $\sigma$ -finiteness conditions will be imposed a priori on  $\mu$ . Then  $L^p(\mu)$ ,  $1 \leq p \leq \infty$ , will denote the usual Banach space associated with  $\Sigma$ -measurable complex-valued functions  $f$  defined on  $X$  with  $\|f\|_p < \infty$ ,  $\|f\|_p$  being the standard  $L^p$ -norm with respect to  $\mu$ . If  $G$  is a locally compact commutative group (always supposed to be Hausdorff),  $L^p(G)$ ,  $1 \leq p \leq \infty$ , will stand for the associated  $L^p$ -space obtained by fixing some Haar (invariant) measure on  $G$ , and  $\widehat{G}$  will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma: G \rightarrow \mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}.$$