# §1. Introduction

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# REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY

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## § 1. INTRODUCTION

A standard version of the Hausdorff-Young inequality for a locally compact commutative group $G$ can be given as follows: for a fixed Haar measure in $G$, let $f \in L^1(G) \cap L^2(G)$; if $1 \leq p \leq 2$, $p' = p/(p-1)$, then

$$\|\widehat{f}\|_{p'} \leq \|f\|_p \tag{1}$$

where

$$\widehat{f}(\gamma) = \int_G f(x)\,\overline{\gamma(x)}\,dx\,, \qquad \gamma \in \widehat{G}\,, \tag{2}$$

$\widehat{G}$ being the dual group of $G$, endowed with a Haar measure which is such that for $p = p' = 2$, there is equality in (1); that this last condition can be met is one form of Plancherel's theorem in $L^2(G)$. Note that, for $1 \leq p \leq 2$, $\|f\|_p < \infty$ if $f$ is in $L^1(G) \cap L^2(G)$, the latter space being dense in each $L^p(G)$, $1 \leq p \leq 2$. Hence, because of the Hausdorff-Young inequality (1), the Fourier transform $\mathcal{F}_p f$ can be defined uniquely for all $f \in L^p(G)$, $1 \leq p \leq 2$, in such a way that

$$\mathcal{F}_p \colon L^p(G) \to L^{p'}(\widehat{G}) \tag{3}$$

is a linear contraction with $\mathcal{F}_p f = \widehat{f}$ for all $f$ in $L^1(G) \cap L^2(G)$. It is known that, for each $p \in [1,2], \mathcal{F}_p$ is injective and that if $f \in L^{p_1}(G) \cap L^{p_2}(G)$, $1 \leq p_1, p_2 \leq 2$, then $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f$ a.e. on $\widehat{G}$; see [HR] vol. 2, chap. VIII ((31.26), p. 229; (31.31), p. 231). The purpose of the present note is to prove

(Thm. 1) by a very simple general argument that the operator $\mathcal{F}_p$ in (3) is surjective only in the following obvious cases: (i) $p = p' = 2$ or (ii) $G$ finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group $G$ whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary $L^p$-spaces (stated in § 2). From this we deduce fairly simply that for any infinite locally compact commutative group $G$, the inequality (1) cannot be extended to the case $2 < p < \infty$; the exact statement is given as Thm. 2 in § 3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in § 2, state and prove the facts alluded to above in § 3 and add a few historical comments in § 4; a short appendix (§ 5) is added to explain the $L^p$-isomorphism theorem stated in § 2.

We have not tried to extend our theorems to the case of $G$ non-commutative, using for $\widehat{G}$ the set of all equivalence classes of continuous unitary irreducible representations of $G$. For $G$ compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in § 4.

## § 2. Notations and some known facts

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple $(X, \Sigma, \mu)$ where $\Sigma$ is a $\sigma$-algebra of subsets of the abstract set $X$ and $\mu \colon \Sigma \to [0, \infty]$ is a $\sigma$-additive positive measure; no finiteness or $\sigma$-finiteness conditions will be imposed a priori on $\mu$. Then $L^p(\mu)$, $1 \leq p \leq \infty$, will denote the usual Banach space associated with $\Sigma$-measurable complex-valued functions $f$ defined on $X$ with $\|f\|_p < \infty$, $\|f\|_p$ being the standard $L^p$-norm with respect to $\mu$. If $G$ is a locally compact commutative group (always supposed to be Hausdorff), $L^p(G)$, $1 \leq p \leq \infty$, will stand for the associated $L^p$-space obtained by fixing some Haar (invariant) measure on $G$, and $\widehat{G}$ will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma \colon G \to \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}.$$