3. A parametric ideal non-symmetric solution of the Tarry-Escott problem of degree four

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3. A PARAMETRIC IDEAL NON-SYMMETRIC SOLUTION OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR

We will now solve the equations

$$a_1 + a_2 + a_3 + a_4 + a_5 = b_1 + b_2 + b_3 + b_4 + b_5 \,, \tag{19}$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = b_1^2 + b_2^2 + b_3^3 + b_4^2 + b_5^2 \,, \tag{20}$$

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = b_1^3 + b_2^3 + b_3^3 + b_4^3 + b_5^3 \,, \tag{21}$$

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 = b_1^4 + b_2^4 + b_3^4 + b_4^4 + b_5^4 \,, \tag{22}$$

so as to get an ideal non-symmetric solution of the Tarry-Escott problem of degree four. We write

$$\begin{aligned} a_1 &= 2px - (\xi + \eta)y \,, & b_1 &= 2px + \eta y, \\ a_2 &= 2qx + \eta y \,, & b_2 &= 2qx - (\xi + \eta)y \,, \\ a_3 &= rx \,, & b_3 &= rx + \zeta y \,, \\ a_4 &= sx - \zeta y \,, & b_4 &= sx \,, \\ a_5 &= \zeta y \,, & b_5 &= -\zeta y \,. \end{aligned} \tag{23}$$

We will first choose p, q, r, s, ξ, η and ζ such that equations (19), (20) and (21) are identically satisfied for all values of x and y. In the equation obtained from (22) by substituting the values of a_i, b_i as given above, the coefficients of x^4 and y^4 on the two sides are equal, and we will choose p, q, r, s, ξ, η and ζ so as to satisfy the additional condition that the coefficient of xy^3 also becomes equal on both sides of this equation. Thus, equation (22) would reduce to an equation containing only the terms x^3y and x^2y^2 and accordingly it can be readily solved for x and y. These values of x and y together with the already suitably chosen values of p, q, r, s, ξ, η and ζ substituted in (23) will give a solution of equations (19), (20), (21) and (22).

When a_i, b_i are defined by (23), we observe that equation (19) is identically satisfied. Substituting the values of a_i, b_i in (20), we note that this equation will also be identically satisfied for all values of x and y if the following condition is satisfied:

$$2(\xi + 2\eta)(p - q) + \zeta(r + s) = 0 \,. \tag{24}$$

Next, we substitute the values of a_i, b_i as given by (23) in equation (21) and observe that the coefficients of x^3 and y^3 on both sides are equal. Equating the coefficients of x^2y and xy^2 on both sides of this equation, we get the following two conditions:

$$4(\xi + 2\eta)(p^2 - q^2) + \zeta(r^2 + s^2) = 0 \,, \tag{25}$$

and

$$2\xi(\xi+2\eta)(p-q)-\zeta^2 r+\zeta^2 s=0\,. \tag{26}$$

Finally, in the equation obtained by substituting the values of a_i, b_i in (22), we equate, as already discussed, the coefficients of xy^3 on both sides to get the additional condition:

$$2(\xi^3+3\xi^2\eta+3\xi\eta^2+2\eta^3)(p-q)+\zeta^3(r+s)=0\,. \tag{27}$$

We now proceed to solve equations (24), (25), (26) and (27). Equations (24) and (27) may be considered as two linear equations in the two linear variables $(p-q)$ and $(r+s)$, and they will be consistent only if ξ, η and ζ satisfy the condition

$$(\xi+2\eta)(\xi^2+\xi\eta+\eta^2-\zeta^2)=0\,.$$

Taking $(\xi + 2\eta) = 0$ leads to trivial solutions, so we will choose ξ, η and ζ such that

$$\xi^2+\xi\eta+\eta^2-\zeta^2=0\,. \tag{28}$$

The complete solution of (28) is readily found to be

$$\xi=2mn-m^2,\qquad \eta=m^2-n^2,\qquad \zeta=m^2-mn+n^2\,. \tag{29}$$

Next, we solve equations (24) and (26) for r and s, and substitute their values in equation (25) which now has a linear factor $(p-q)$ that can be ignored and then equation (25) is readily seen to be satisfied if we choose p and q as follows:

$$\begin{aligned} p&=\xi^3+2\xi^2\eta+\xi\zeta^2+2\eta\zeta^2-2\zeta^3\,,\\ q&=\xi^3+2\xi^2\eta+\xi\zeta^2+2\eta\zeta^2+2\zeta^3\,. \end{aligned} \tag{30}$$

With these values of p and q, we immediately get

$$\begin{aligned} r&=-4\xi^2\zeta-8\xi\eta\zeta+4\xi\zeta^2+8\eta\zeta^2\,,\\ s&=4\xi^2\zeta+8\xi\eta\zeta+4\xi\zeta^2+8\eta\zeta^2\,. \end{aligned} \tag{31}$$

Thus, when ξ, η, ζ are defined by (29), and p, q, r and s are given by (30) and (31), equations (24), (25), (26) and (27) are all satisfied. With these values of p, q, r, s, ξ, η and ζ, equation (22) reduces, on removing the factor $64x^2y\zeta^3(\xi+2\eta)$, to

$$\begin{aligned}(6\xi^6+24\xi^5\eta+24\xi^4\eta^2-12\xi^4\zeta^2-48\xi^3\eta\zeta^2-48\xi^2\eta^2\zeta^2-2\xi^2\zeta^4\\ -8\xi\eta\zeta^4-8\eta^2\zeta^4+8\zeta^6)x-(3\xi^4+6\xi^3\eta-3\xi^2\zeta^2-6\xi\eta\zeta^2)y=0\,.\end{aligned}$$

Thus, equation (22) will be satisfied if we choose

$$
\begin{aligned}
x &= 3\xi^4 + 6\xi^3\eta - 3\xi^2\zeta^2 - 6\xi\eta\zeta^2, \\
y &= 6\xi^6 + 24\xi^5\eta + 24\xi^4\eta^2 - 12\xi^4\zeta^2 - 48\xi^3\eta\zeta^2 \\
&\quad - 48\xi^2\eta^2\zeta^2 - 2\xi^2\zeta^4 - 8\xi\eta\zeta^4 - 8\eta^2\zeta^4 + 8\zeta^6 .
\end{aligned}
\tag{32}
$$

A solution of equations (19), (20), (21) and (22) can now be obtained in terms of the parameters m and n by taking ξ, η and ζ as given by (29), substituting these values of ξ, η and ζ in (30), (31) and (32) to obtain p, q, r, s, x and y in terms of m and n, and then substituting the values of p, q, r, s, ξ, η, ζ, x and y in (23). The solution so obtained may, after simplification and removal of common factors, be written explicitly in terms of the arbitrary parameters m and n as follows:

$$
\begin{aligned}
a_1 &= 12m^7n - 37m^6n^2 + 24m^5n^3 + 12m^4n^4 - 20m^3n^5 \\
&\quad + 15m^2n^6 - 18mn^7 + 8n^8, \\
a_2 &= 10m^7n - 30m^6n^2 + 54m^5n^3 - 13m^4n^4 - 48m^3n^5 \\
&\quad + 45m^2n^6 - 14mn^7, \\
a_3 &= 4m^8 + 6m^7n - 28m^6n^2 + 8m^5n^3 + 66m^4n^4 \\
&\quad - 128m^3n^5 + 112m^2n^6 - 48mn^7 + 8n^8, \\
a_4 &= 4m^8 - 12m^7n + 35m^6n^2 - 55m^5n^3 + 66m^4n^4 \\
&\quad - 65m^3n^5 + 49m^2n^6 - 30mn^7 + 8n^8, \\
a_5 &= -4m^8 + 14m^7n - 27m^6n^2 + 55m^5n^3 - 80m^4n^4 \\
&\quad + 81m^3n^5 - 49m^2n^6 + 14mn^7, \\
b_1 &= -4m^8 + 14m^7n - 22m^6n^2 + 10m^5n^3 + 67m^4n^4 \\
&\quad - 140m^3n^5 + 113m^2n^6 - 46mn^7 + 8n^8, \\
b_2 &= 4m^8 + 8m^7n - 45m^6n^2 + 68m^5n^3 - 68m^4n^4 \\
&\quad + 72m^3n^5 - 53m^2n^6 + 14mn^7, \\
b_3 &= 20m^7n - 55m^6n^2 + 63m^5n^3 - 14m^4n^4 - 47m^3n^5 \\
&\quad + 63m^2n^6 - 34mn^7 + 8n^8, \\
b_4 &= 2m^7n + 8m^6n^2 - 14m^4n^4 + 16m^3n^5 - 16mn^7 + 8n^8, \\
b_5 &= 4m^8 - 14m^7n + 27m^6n^2 - 55m^5n^3 + 80m^4n^4 \\
&\quad - 81m^3n^5 + 49m^2n^6 - 14mn^7 .
\end{aligned}
\tag{33}
$$

We may apply Frolov's theorem to the above solution to obtain other non-symmetric solutions. For instance, an arbitrary constant K can be added to all the terms a_i, b_i, $i = 1, 2, 3, 4, 5$.

As a numerical example, taking $m = 3$, $n = 1$, we get, on suitable re-arrangement, the following solution:

$$\begin{aligned}(-1659)^r + 1406^r + 2784^r + 4025^r + 5915^r \\ = (-1675)^r + 1659^r + 2366^r + 4256^r + 5865^r,\end{aligned}$$

where $r = 1, 2, 3, 4$. Adding the constant 1676 to all the terms, we get the following solution in positive integers:

$$17^r + 3082^r + 4460^r + 5701^r + 7591^r = 1^r + 3335^r + 4042^r + 5932^r + 7541^r,$$

where $r = 1, 2, 3, 4$.

4. THE DIOPHANTINE SYSTEM $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4, 6$

We will now state the theorem given by Gloden [7, p. 24] to which a reference has already been made in the introduction and then apply it to obtain a parametric solution of this diophantine system.

THEOREM 4.1. *If*

$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \qquad r = 1, 2, \ldots, k$$

then

$$\sum_{i=1}^{k+1} (a_i + t)^r = \sum_{i=1}^{k+1} (b_i + t)^r, \qquad r = 1, 2, \ldots, k, k+2,$$

where

$$t = -\left(\sum_{i=1}^{k+1} a_i\right)/(k+1).$$

As we have already obtained, in the preceding section, a parametric solution of $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4$, a direct application of the above theorem gives a parametric solution of $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4$ and 6. We multiply the $(a_i + t)$, $(b_i + t)$, $i = 1, 2, 3, 4, 5$ by 5 to cancel out denominators,