

## 6. Applications

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## 6. APPLICATIONS

LEMMA 13. *Let  $M$  be a closed hyperbolic surface of genus  $g$  which has  $2g - 2$  simple closed geodesics  $u_1, \dots, u_{2g-2}$  which all intersect in the same point  $Q$  and intersect in no other point. Then  $M$  has simple closed curves  $u_{2g-1}$  and  $u_{2g}$ , passing through  $Q$ , such that the curves  $u_i$  intersect in no other point than  $Q$ ,  $i = 1, \dots, 2g$ . Moreover,  $u_{2g-1}$  and  $u_g$  can be chosen such that*

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

*is the interior of a canonical polygon  $P(g)$ .*

*Proof.* Cut  $M$  along  $u_1$ , the result is a hyperbolic surface  $M_1$  with boundary and genus  $g - 1$ , the boundary consists of two simple closed geodesics  $v_1$  and  $w_1$ . Cut  $M_1$  along  $u_2$ , the result is a hyperbolic surface  $M_2$  with one boundary component  $v_2$  and genus  $g - 1$ . Now cut  $M$  along all  $2g - 2$  simple closed geodesics  $u_1, \dots, u_{2g-2}$ . By induction, the result is a hyperbolic surface  $M_{2g-2}$  with one boundary component  $v$  and genus 1. More precisely, the boundary  $v$  is piecewise geodesic with  $4g - 4$  pieces and we may assume that the notation is chosen such that these pieces appear on  $v$  in the order (the pieces are called like the corresponding closed curves)  $u_1, u_2, \dots, u_{2g-2}, u_1, u_2, \dots, u_{2g-2}$  (note that closed geodesics intersect transversally). Denote by  $S$  and  $S'$  the two copies of  $Q$  on  $v$  between  $u_1$  and  $u_{2g-2}$ . Let  $u_{2g-1}$  be a simple geodesic in  $M_{2g-2}$  which joins  $S$  and  $S'$  such that  $u_{2g-1}$  is not homotopic to a part of  $v$ . Cut  $M_{2g-2}$  along  $u_{2g-1}$ . The result is a hyperbolic surface  $M_{2g-1}$  of genus zero with two boundary components  $w$  and  $w'$  which both consist of  $2g - 1$  geodesic pieces in the order  $u_1, u_2, \dots, u_{2g-2}, u_{2g-1}$ . Denote by  $R$  and  $R'$  the copies of  $Q$  between  $u_1$  and  $u_{2g-1}$  on  $w$  and  $w'$ , respectively. Let  $u_{2g}$  be a simple geodesic in  $M_{2g-1}$  which joins  $R$  and  $R'$ ,  $u_{2g}$  can be chosen such that when we cut  $M_{2g-1}$  along  $u_{2g}$ , then we obtain the interior of a canonical polygon as desired.  $\square$

DEFINITION. A *hyperelliptic surface* is a closed hyperbolic surface of genus  $g$  which has an isometry  $\phi$  with  $\phi^2 = id$  and with exactly  $2g + 2$  fixed points.

In [14], the equivalence of (i) and (ii) of the following theorem was first proved. With the approach chosen here, we can give a third equivalence and

a different proof.

**THEOREM 14.** *Let  $M$  be a closed hyperbolic surface  $M$  of genus  $g$ . Then the following conditions are equivalent.*

- (i)  $M$  is hyperelliptic.
- (ii)  $M$  has a set of at least  $2g-2$  simple closed geodesics which all intersect in the same point and intersect in no other point.
- (iii)  $M$  has a corresponding canonical polygon with equal opposite angles ( $\alpha_i = \alpha_{2g+i}$ ,  $i = 1, \dots, 2g$ ).

*Proof.* I shall prove (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i).

Let  $M$  be hyperelliptic. Let  $R_i$ ,  $i = 1, \dots, 2g+2$ , be the fixed points of a hyperelliptic involution  $\phi$ . Let  $c_1$  be a simple geodesic segment from  $R_1$  to  $R_2$ . Then  $c_1 \cup \phi(c_1)$  is a simple closed geodesic  $u_1$  since  $\phi^2 = id$ . It also follows that on  $u_1$ , there are only two fixed points of  $\phi$  and that  $M_1 = M \setminus u_1$  is connected. Therefore, we can choose a simple geodesic segment  $c_2$  from  $R_1$  to  $R_3$  which intersects  $u_1$  only in  $R_1$ . By the same argument as above,  $c_2 \cup \phi(c_2)$  is a simple closed geodesic,  $M_2 = M \setminus (u_1 \cup u_2)$  is connected and on  $u_1 \cup u_2$ , there are only three fixed points of  $\phi$ . Continuing this construction we can find simple closed geodesics  $u_1, \dots, u_{2g-2}$  which all intersect in  $R_1$  and in no other point. This proves (i)  $\Rightarrow$  (ii).

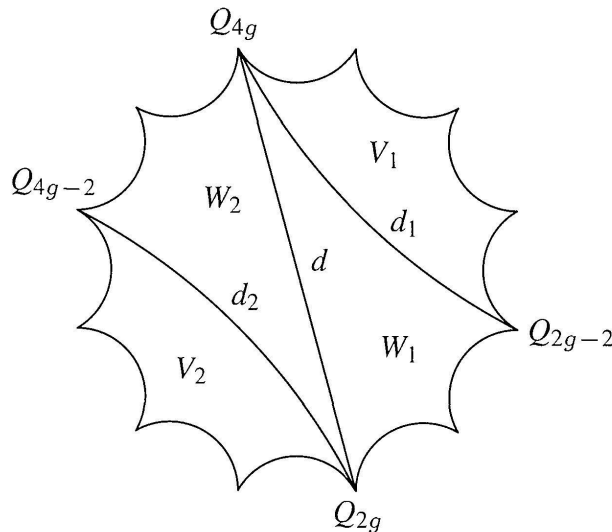


FIGURE 6

The partition of a canonical polygon  $P(g)$  into two  $(2g-1)$ -gons and two quadrilaterals

Assume now that  $M$  has  $2g-2$  simple closed geodesics  $u_1, \dots, u_{2g-2}$  which all intersect in the same point  $Q$  and intersect in no other point. By Lemma 13 we then can find simple closed curves  $u_{2g-1}$  and  $u_{2g}$  such that

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon  $P(g)$  with the usual notation. For  $i = 1, \dots, 4g$ , let  $\{Q_i\} = a_i \cap a_{i+1}$ . In  $P(g)$  let  $d_1$  be the geodesic segment from  $Q_{4g}$  to  $Q_{2g-2}$ ,  $d_2$  the geodesic segment from  $Q_{2g}$  to  $Q_{4g-2}$ , and  $d$  the geodesic segment from  $Q_{2g}$  to  $Q_{4g}$ , compare Figure 6. Then  $P(g) \setminus (d_1 \cup d_2 \cup d)$  has four connected components, two quadrilaterals  $W_j$  having  $d$  and  $d_j$ ,  $j = 1, 2$ , among the sides and two  $(2g - 1)$ -gons  $V_j$  having  $d_j$  among the sides,  $j = 1, 2$ . Since  $u_i$ ,  $i = 1, \dots, 2g - 2$ , are simple closed geodesics, it follows that  $\alpha_i = \alpha_{i+2g}$  for  $i = 1, \dots, 2g - 3$ . This implies that  $V_1$  and  $V_2$  are isometric and that  $d_1$  and  $d_2$  have the same length. Therefore,  $W_1$  and  $W_2$  are quadrilaterals with equal lengths of the four sides. Fix now  $W_1$  and try to vary  $W_2$  such that the lengths of the sides remain invariant and so that property (V) for canonical polygons holds. This is certainly the case if  $W_2$  and  $W_1$  are isometric. But then Corollary 8 implies that this is the unique possibility. Therefore,  $W_1$  and  $W_2$  must be isometric and hence  $\alpha_i = \alpha_{i+2g}$  for all  $i = 1, \dots, 2g$ , which proves (ii)  $\Rightarrow$  (iii).

Now assume that (iii) holds. Let  $d$  be the geodesic segment from  $Q_{2g}$  to  $Q_{4g}$ . Then  $d$  separates  $P(g)$  into two isometric  $(2g + 1)$ -gons and the  $\pi$ -rotation around the centre  $C$  of  $d$  induces an isometry  $\phi$  of  $M$  with  $\phi^2 = id$ . The fixed points of  $\phi$  are  $C$ , the point  $Q$  corresponding to the vertices of  $P(g)$  as well as the centres of the sides  $a_i$ ,  $i = 1, \dots, 2g$ . Therefore,  $\phi$  is a hyperelliptic involution which proves (iii)  $\Rightarrow$  (i).  $\square$

**COROLLARY 15.** *All closed hyperbolic surfaces of genus 2 are hyperelliptic.*

*Proof.* All closed hyperbolic surfaces have two simple closed geodesics which intersect in a unique point. The corollary follows by Theorem 14.  $\square$

**DEFINITION.** Let  $M_0$  be a closed hyperbolic surface in  $T_g$ . For every  $M \in T_g$  fix a homeomorphism  $\phi_M$ , homotopic to the identity, from  $M_0$  to  $M$  ( $\phi_M$  exists since closed surfaces of the same genus are homeomorphic). Let  $u$  be a simple closed geodesic in  $M_0$ . Then, in the homotopy class of  $\phi_M(u)$  there exists a unique simple closed geodesic which is denoted by  $\Phi_M(u)$ . The function

$$L(u): T_g \rightarrow \mathbf{R}$$

which associates to  $M$  the length of  $\Phi_M(u)$  is called a *geodesic length function*.

REMARK. It is well known that  $T_g$  can be parametrized by a finite number of geodesic length functions, see for example [12], [13] where it is shown that  $T_g$  can be parametrized by  $6g - 5$  geodesic length functions.

THEOREM 16. *The Teichmüller space  $T_g$  for  $g = 2$  can be parametrized by 7 (suitably chosen) geodesic length functions  $L(u_1), \dots, L(u_7)$ , taken as homogeneous parameters (which means that  $L(u_1)/L(u_7), \dots, L(u_6)/L(u_7)$  gives a parametrization of  $T_2$ ).*

*Proof.* Let  $P(2)$  be a canonical polygon corresponding to a closed hyperbolic surface  $M_0$  of genus 2. As usual let  $Q_i = a_i \cap a_{i+1}$ ,  $i = 1, \dots, 8$ , where the  $a_i$  are the sides of  $P(2)$ . Let  $b_i$  be the geodesic segment (in  $P(2)$ ) between  $Q_i$  and  $Q_{i+4}$ ,  $i = 1, \dots, 4$ . By Corollary 15,  $M_0$  is hyperelliptic, therefore (compare Theorem 14)  $b_i$  corresponds to a simple closed geodesic in  $M_0$ , denoted by  $B_i$ ,  $i = 1, \dots, 4$ . It also follows by Theorem 14 that  $a_i$  corresponds to a simple closed geodesic in  $M_0$ , denoted by  $A_i$ ,  $i = 1, \dots, 4$ .

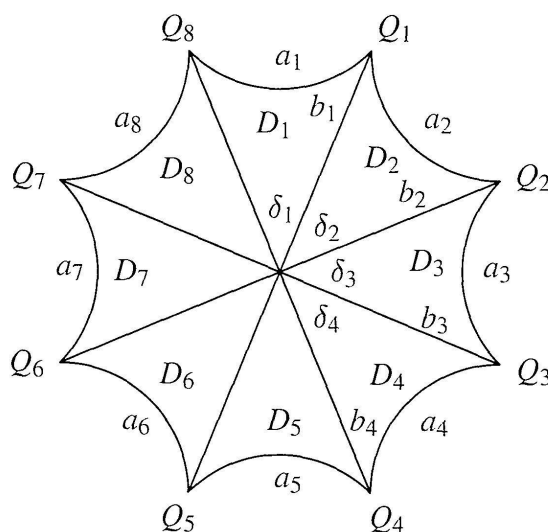


FIGURE 7

A triangulation of a canonical polygon  $P(g)$  for  $g = 2$

I now prove that the 7 length functions, given by the simple closed geodesics  $A_i$ ,  $i = 1, 2, 3$ ,  $B_i$ ,  $i = 1, \dots, 4$ , taken as homogeneous parameters, give a parametrization of  $T_2$ . In order to do this, it is enough (by Theorem 11 and Corollary 12) to show that  $P(2)$  is uniquely determined by the lengths of  $a_i$ ,  $i = 1, 2, 3$ ,  $b_i$ ,  $i = 1, \dots, 4$ , taken as homogeneous parameters (in the sequel I shall refer to these lengths calling them “the seven lengths”). This can be done analogously as in the proof of Theorem 11. The geodesic segments  $b_i$ ,  $i = 1, \dots, 4$ , intersect in a point  $C$ , the “centre” of  $P(2)$ , and they separate

$P(2)$  into 8 triangles  $D_j$  so that  $a_j$  is a side of  $D_j$ ,  $j = 1, \dots, 8$ , compare Figure 7. Since  $M$  is hyperelliptic,  $D_j$  and  $D_{j+4}$  are isometric,  $j = 1, \dots, 4$ . Denote by  $\delta_i$  the angle of  $D_i$  in the vertex  $C$ ,  $i = 1, \dots, 4$ . The seven lengths determine the triangles  $D_i$ ,  $i = 1, 2, 3$ , as well as two sides and the angle  $\delta_4$  of  $D_4$  by the condition

$$(6) \quad \Delta := \sum_{j=1}^4 \delta_j = \pi,$$

so they determine also  $D_4$ . This shows that the seven lengths determine  $P(2)$ . Multiply the seven lengths by a positive real  $t$  and assume that the seven new lengths also determine a canonical polygon  $P_t(2)$ . If  $t > 1$ , then  $\delta_i$ ,  $i = 1, 2, 3$ , are smaller in  $P_t(2)$  than in  $P(2)$  by Lemma 9, therefore, by (6),  $\delta_4$  is larger in  $P_t(2)$  than in  $P(2)$ . It follows by Lemma 7 that the sum of the two other angles of  $D_4$  is smaller in  $P_t(2)$  than in  $P(2)$ . Since all angles in  $D_i$ ,  $i = 1, 2, 3$ , are smaller in  $P_t(2)$  than in  $P(2)$  by Lemma 9, it follows that

$$\sum_{i=1}^4 \alpha_i$$

is smaller in  $P_t(2)$  than in  $P(2)$ . But this contradicts condition (II) of canonical polygons. An analogous contradiction follows if  $t < 1$  proving thus that  $t = 1$  and therefore the theorem.  $\square$

REMARK. Theorem 16 is new. It is well known that  $6g-6$  length functions can never parametrize  $T_g$  so that the situation of Theorem 16 is the best we can expect. It is not known whether  $6g-5$  geodesic length functions, *taken as homogeneous parameters*, can parametrize  $T_g$  for  $g \geq 3$ .

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