

## 5. Parallel transport in a cubical manifold and the proof of theorem 3

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the sequence of curves  $r \circ \gamma_m$  converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics  $\gamma_m$  converges locally uniformly. By definition, this means that the corresponding subsequence of  $(x_m)$  converges to a point  $\xi \in X(\infty)$ .

Let  $\phi \in \Gamma$  and choose  $c = c_\phi$  as in Lemma 4.8. Let  $t_0 > 0$  be given. By Lemma 4.8 we have  $r \circ \gamma_m(t_0) \in F$  for all  $m \geq m_0$ . By Proposition 3.4 and Lemma 4.8, we have  $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n}c_\phi$  for all such  $m$ . Now  $c_\phi$  is independent of  $t_0$ , hence  $\phi(\xi) = \xi$ .  $\square$

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1,  $\Delta \cong \ker h$  consists precisely of the elliptic elements of  $\Gamma$ . If indices of type 1 do not occur, then Proposition 4.3 applies: If  $k = 0$ , then  $\Gamma \cong \Delta$  fixes a point of  $X$  and possibility (1) holds. If  $k > 0$ , then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that  $\text{Stab}(x) \neq \Delta$  for any  $x \in X$  in this case since  $\Delta$  would have a fixed point otherwise.

## 5. PARALLEL TRANSPORT IN A CUBICAL MANIFOLD AND THE PROOF OF THEOREM 3

Let  $X$  be a cubical manifold of dimension  $n$ . Given two chambers  $P$  and  $Q$  in  $X$  with a common face of dimension  $n - 1$ , we define  $t_{PQ}: P \rightarrow Q$  to be the *translation* which moves each point  $p$  of  $P$  along the unit geodesic segment starting at  $p$  and orthogonal to the common  $(n - 1)$ -face of  $P$  to the end point in  $Q$ . The map  $t_{PQ}$  is an isomorphism and isometry of  $P$  with  $Q$ . Given a gallery  $\pi = (P_1, \dots, P_n)$  in  $X$ , the *parallel transport* along  $\pi$  is the isomorphism  $t_\pi: P_1 \rightarrow P_n$  given by

$$t_\pi := t_{P_{n-1}P_n} \circ \dots \circ t_{P_2P_3} \circ t_{P_1P_2}.$$

**LEMMA 5.1.** *Let  $X$  be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in  $X$  is divisible by 4. Then for any two chambers  $P$  and  $Q$  in  $X$ , the parallel transport  $t_\pi$  along a gallery  $\pi$  connecting  $P$  and  $Q$  is independent of  $\pi$ .*

*Proof.* It is enough to show that the parallel transport along any closed gallery is the identity. Let  $\pi$  be such a gallery with initial and final chamber  $P$ .

Represent  $\pi$  by a closed curve  $c$  which starts and ends in some interior point  $p$  of  $P$ , such that  $c$  misses the  $(n-2)$ -skeleton of  $X$  and crosses  $(n-1)$ -faces transversally and according to the pattern provided by  $\pi$ . Since  $X$  is simply connected, the curve  $c$  can be contracted in  $X$  to the point  $p$ . Since  $X$  is a manifold, the links of the vertices in  $X$  are  $(n-2)$ -connected. Hence the contraction of  $c$  can be chosen to be generic in the sense that it misses the  $(n-3)$ -skeleton of  $X$  and crosses the  $(n-2)$ -skeleton transversally. Following the curve  $c$  along this contraction, we get a sequence of modifications of the gallery  $\pi$ . These modifications occur when  $c$  crosses an  $(n-2)$ -face of  $X$ . The condition that the number of chambers adjacent to such faces is divisible by 4 implies that the parallel transport  $t_\pi$  does not change under these modifications. Since the parallel transport along the trivial gallery is the identity,  $t_\pi = \text{id}_P$ .  $\square$

From now on we assume that  $X$  is a simply connected cubical manifold such that the number of chambers adjacent to each face of codimension 2 in  $X$  is divisible by 4. For chambers  $P$  and  $Q$  in  $X$  define  $t_{PQ} = t_\pi$ , where  $\pi$  is any gallery connecting  $P$  with  $Q$ . The above lemma shows that  $t_{PQ}$  is well defined.

We fix a chamber  $P_0$  of  $X$  and define a homomorphism  $\phi: \Gamma \rightarrow \text{Aut } P_0$  by

$$\phi(g) := t_{g(P_0)P_0} \circ g|_{P_0}.$$

The kernel  $\Gamma' := \ker \phi$  is a finite index subgroup of  $\Gamma$  and consists precisely of those automorphisms of  $\Gamma$  whose restriction to any chamber commutes with the corresponding parallel transport.

## COORIENTATIONS

A *coorientation of a wall* in a chamber is a choice of one of the two half-chambers determined by the wall. Once and for all, we choose coorientations of the walls in the above chamber  $P_0$ . Now by Lemma 5.1, parallel transport gives rise to a consistent choice of coorientations for all walls in  $X$ .

By Corollary 1.3,  $X$  is foldable. We fix a folding and denote by  $\Lambda_i$  the set of hyperspaces of  $X$  with label  $i$ . Note that  $\Lambda_i$  is invariant under parallel transport. Along a hyperspace with label  $i$ , the half-chambers distinguished by the coorientation are all contained in the same halfspace with respect to the hyperspace. The above group  $\Gamma'$  preserves the families  $\Lambda_i$  together with the coorientations.

The index of intersection of an oriented curve  $c$  at a transversal crossing of a hyperspace  $H \in \Lambda_i$  is defined to be equal to  $+1$  or  $-1$  respectively, according to whether the orientation of  $c$  coincides with the coorientation of  $H$  or not. Fix a point  $p_0$  in the interior of  $P_0$  which does not belong to any wall and any of the chosen coorientations. For  $p \in X$  define  $f_i(p)$  to be the sum of the indices of intersection of an oriented curve  $c$  connecting  $p_0$  and  $p$  with the hyperspaces from  $\Lambda_i$ . Here we assume that  $c$  is generic, i.e.  $c$  does not meet the  $(n-2)$ -skeleton and crosses hyperspaces transversally. The integer  $f_i(p)$  does not depend on  $c$  since  $X$  is simply connected and any two such curves can be deformed into each other by a homotopy which misses the  $(n-3)$ -skeleton of  $X$  and crosses the  $(n-2)$ -skeleton of  $X$  transversally.

For  $g \in \Gamma$  set  $h_i(g) = f_i(g(p_0))$ . Since the chosen system of coorientations is invariant under the action of  $\Gamma'$ , the maps  $h_i: \Gamma' \rightarrow \mathbf{Z}$  are homomorphisms. We finish the proof of Theorem 3 by showing that the image of  $h = (h_1, \dots, h_n)$  is of finite index in  $\mathbf{Z}^n$ .

We need to show that the image of  $h$  contains  $n$  linearly independent vectors. To that end, we show that the image contains non-zero vectors which span arbitrarily small angles with the unit vectors  $e_i$  in  $\mathbf{R}^n$ ,  $1 \leq i \leq n$ . Let  $\sigma$  be a unit speed geodesic ray with  $\sigma(0) = p_0$  which is perpendicular in  $P_0$  to the wall with label  $i$ . By the choice of  $p_0$ , the ray  $\sigma$  does not meet the  $(n-2)$ -skeleton of  $X$  and is perpendicular to all  $(n-1)$ -faces and walls with label  $i$  which it intersects. We have

$$f_j(\sigma(m)) = \delta_{ij} \cdot m, \quad m \geq 1.$$

By the cocompactness of the action of  $\Gamma'$ , there is an integer  $k \geq 1$  such that for any  $m \geq 1$  there is a  $g_m \in \Gamma'$  with  $d(\sigma(m), g_m(p_0)) \leq k$ . By the definition of  $h_j$  this implies  $|h_j(g_m) - f_j(\sigma(m))| \leq k$ . Theorem 3 follows.