5. Proof of the Theorem

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In this section we prove the theorem stated in the Introduction. The principle of the proof is given in the following remark.

REMARK 7. *Let* $G = \mathrm{Aut}(F) \leq GL_d(\mathbf{Z})$ *be a uniform automorphism group of the lattice* $L = \mathbf{Z}^d$ *and let* $n \in N$, *where* N *is the normaliser of* G. *Then by Remark 1,* n *induces a similarity* $L \to Ln$. *Remark 2 says that the lattice* Ln *is also* G*-invariant. Let* $L' \in \pi(L)$ *such that* $\det(n) = [L : (L \cap Ln)]\,[Ln : (L \cap Ln)]^{-1}$ *equals* $[L' : L]^{-1}$. *Then* $[L' : (L' \cap Ln)] = [Ln : (L' \cap Ln)]$. *So one may conclude that, if there is no other* G*-invariant lattice* M *in the layer of* L' *(i.e. with* $[L' : (M \cap L')] = [M : (M \cap L')]$ *), then* $Ln = L'$.

The last uniqueness condition is fulfilled if $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$, all lattices in $\mathcal{Z}(G)$ are even, and G is lattice sparse according to the following definition. In this case $\mathcal{Z}(G) = \{aL' \mid L' \in \pi(L),\ a \in \mathbf{Q}^*\}$ for any G-invariant lattice L. Note that this implies that the exponent $\exp(L^\#/L)$ is square free.

DEFINITION 8. *If* p *is a prime then a finite group* $G \leq GL_d(\mathbf{Q})$ *is called* p-lattice sparse *if any lattice* $L \in \mathcal{Z}(G)$ *can be obtained from any other lattice in* $\mathcal{Z}(G)$ *that contains* L *of* p*-power index by a combination of the five operations: taking sums, intersections, the dual lattice or the even sublattice with respect to some* $F \in \mathcal{F}_{>0}(G)$, *or multiplying by invertible elements of* $C_{M_d(\mathbf{Q})}(G)$. *The group* G *is called* lattice sparse *if* G *is* p*-lattice sparse for all primes* p.

Since the proof of the Theorem is similar for all r.i.m.f. groups, we only deal with the most interesting cases $d = 16$ and 24.

The r.i.m.f. groups of degree 16 and 24 fixing strongly modular lattices, which are not proper tensor products, are displayed in the table. The first column gives the number of the group under which it is referred to in [NeP 95] or [Neb 95] and [Neb 96]. The second column contains a name for the matrix group as partially explained in the paragraph preceding Proposition 5. In the notation there we additional make the following abbreviations. If $z = d_1$ or d_2, we omit $\times$ and Q in the symbols. Also (1) is omitted if $p = 1$. The division algebra Q is abbreviated as α, if $Q = \mathbf{Q}[\alpha]$, by the set of ramified primes, if Q is a quaternion algebra over $\mathbf{Q}$, and omitted if $Q = \mathbf{Q}$. For the finite simple and quasisimple groups we use the notation of [CCNPW 85] except that the alternating group is denoted by Alt_n to avoid confusion with A_n, which also denotes the automorphism group of the root lattice A_n.

The next three columns give information about the invariant strongly modular lattice L. First the determinant $\det(L)$ is given as the product of the abelian invariants of the Sylow subgroups of $L^{\#}/L$. The next column contains $\min(L)$ the minimum of the square lengths of the non zero vectors in L. From these two columns one may see whether L is an extremal lattice as defined in [Que 96]. The number of vectors of square length $\min(L)$ decomposed as a sum of the orbit lengths under $\mathrm{Aut}(L)$ is displayed in the fifth column. The last column contains the primes p for which $\mathrm{Aut}(L)$ is p-lattice sparse. A + indicates that $\mathrm{Aut}(L)$ is lattice sparse.

Proof of the Theorem. The commuting algebras of the groups in the table are all isomorphic to $\mathbf{Q}$ except for the one of $[6.\mathrm{Alt}_7 : 2]_{24}$ which is $\mathbf{Q}[\sqrt{-6}]$. So all these groups are uniform. Since the arguments are similar for all groups G we only deal with $G = \left[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} SL_2(9)\right]_{16}$ extensively. Let $L \in \mathcal{Z}(G)$ be a G-invariant lattice. There is a unique $F \in \mathcal{F}_{>0}(L)$ such that $\{l_1 F l_2^{tr} \mid l_1, l_2 \in L\} = \mathbf{Z}$. The determinant of L with respect to F is $|L^{\#}/L| = 3^8 \cdot 5^8$ and its minimum is 10. If this lattice is strongly modular, then it is an extremal strongly modular lattice ([Que 96]).

Since G is of the form described in Proposition 5 with $p = 3$, there is an element $a_1 \in N = N_{GL_{16}(\mathbf{Q})}(G)$ with $\frac{1}{3}a_1^2 \in G$. Since G is lattice sparse and $\det(a_1) = \frac{1}{3}^8$ one has $La_1 = 3L^{\#} \cap L \in \mathcal{Z}(G)$ and $a_1 F a_1^{tr} = 3F$ (by Remarks 2 and 1). Hence a_1 induces a similarity between L and $3L^{\#} \cap L$.

Next consider the normal subgroup $U := SL_2(5) \underset{\infty,3}{\otimes} SL_2(9) \trianglelefteq G$. The commuting algebra $C_{M_{16}(\mathbf{Q})}(U) =: K$ is isomorphic to $\mathbf{Q}[\sqrt{5}]$. From Proposition 4 one obtains an element $c \in N$ with $c^2 = 5$. As above one concludes that c yields a similarity between L and $5L^{\#} \cap L$. The product $a_1 c \in N$ is of determinant $\pm 15^8$ and gives a similarity between L and $15L^{\#}$. Therefore L is strongly modular.

Most of the other groups can be dealt with similarly. One has to use Proposition 6 to construct an additional element of N for the r.i.m.f. groups 4 and 14 of $GL_{16}(\mathbf{Q})$. For $G = [2.\mathrm{Alt}_{10}]_{16}$ (number 14), one obtains $n \in N$ of determinant $\pm 5^8$, since the character extends to $2.S_{10}$ with character field $\mathbf{Q}[\sqrt{\pm 5}]$ (cf. [CCNPW 85]). Analogously, for $F_4 \tilde{\otimes} F_4 = 2_+^{1+8}.O_8^+(2)$ (number 4) the character extends to $2_+^{1+8}.O_8^+(2).2$ with character field $\mathbf{Q}[\sqrt{\pm 2}]$.

The strong modularity for the lattices of the r.i.m.f. groups 9 and 21 of $GL_{16}(\mathbf{Q})$ (in particular the similarity of L with the lattice corresponding to the Sylow-2-subgroup of $L^{\#}/L$) may be derived from the equality

$$\left[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)\right]_{16} = \left[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)\right]_{16}$$

and

$$\left[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)\right]_{16} = \left[(SL_2(5).2 \circ C_3) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)\right]_{16}$$

using Proposition 5.

Similarly one uses Proposition 5 to show the 2-modularity of the lattices of the r.i.m.f. group 6 in $GL_{24}(\mathbf{Q})$ using the description

$$\left[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)\right]_{24} = \left[6.U_4(3).2 \overset{2(2)}{\circ} SL_2(3)\right]_{24}.$$

For the groups 44 and 64, which are the only groups which are not p-lattice sparse for a relevant prime p (=2), one has to note that the invariant sublattice of index 2^{12} in L is unique.

The theorem now follows from the next lemma. □

LEMMA 9. *The lattices (of determinant* $3^8 \cdot 5^8$*) of the r.i.m.f. subgroup* $G := [\pm \mathrm{Alt}_6 .2^2]_{16} \leq GL_{16}(\mathbf{Q})$ *(number 20 of* [NeP 95]*) are not (strongly) modular.*

Proof. Let L be such a G-invariant lattice and $L' \in \pi(L)$. Assume that there is a similarity $s : L' \to L$. By Proposition 3, this similarity s normalises G. Let $U \cong \mathrm{Alt}_6$ be the characteristic subgroup $\cong \mathrm{Alt}_6$ of G. Since the full automorphism group of U is already induced by conjugation with elements of G, there exists $g \in G$, such that $n := gs \in GL_{16}(\mathbf{Q})$ centralises U. Hence $n \in C_{M_{16}(\mathbf{Q})}(U) \cong \mathbf{Q}[\sqrt{5}]$. Since this number field does not contain an element of norm 3, one concludes that $[L' : L] = 5^8$. So the lattice L is neither similar to $L^{\#}$ nor to the lattice $L' \in \pi(L)$ corresponding to the 3-Sylow subgroup of $L^{\#}/L$. Note that if $[L' : L] = 5^8$, an element $x \in C_{M_{16}(\mathbf{Q})}(U)$ with $x^2 = 5$, induces a similarity by Proposition 4. □

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