7. A RECTANGULAR LATTICE

To satisfy (3.11) we *redefine*

$$\pi_i(t) = a\mathscr{P}(t - c_i/2) , \tag{6.18}$$

and set

$$G_0(z_1, z_2) = a^4 G(z_1/a, z_2/a) , \tag{6.19}$$

so that

$$\overline{G_0(z_1, z_2)} = G_0(\bar{z}_2, \bar{z}_1) . \tag{6.20}$$

In summary we have

PROPOSITION 6.1. *Let* $\Lambda = \mathbf{C}/\Lambda$ *have the holomorphic involutions (6.1) intertwined by the anti-holomorphic involution (5.6). Then* (Γ, ρ, τ_i) *is realized by the map (6.5), (6.18) onto the quartic curve* $G_0(z_1, z_2) = 0$ *given by (6.13), (6.14), (6.19). If the fixed-point set of* ρ *is non-empty, then this is the complexification of the real curve* $G_0(z, \bar{z}) = 0$.

7. A RECTANGULAR LATTICE

We consider the special case of Λ, ρ, τ_i as given in (5.19), (5.6), (6.1), with

$$a = +1, b = 0, \bar{c}_2 = c_1 = c_1' + ic_1'', c = ic_1'' . \tag{7.1}$$

From (6.16), (6.15) it follows that g_2, g_3, β are real, and β' is purely imaginary. Thus, the coefficients $\beta_1, \beta_2, \beta_3$ of $G(z_1, z_2)$ are real. With $t = t' + it''$, we have

$$FP(\rho) = \{t'' = 0\} \cup \{t'' = \omega''/2\} , \tag{7.2}$$

$$\tau_1\{t'' = 0\} = \{t'' = c_1''\}, \ \tau_1\{t'' = \omega''/2\} = \{t'' = c_1'' + \omega''/2\} . \tag{7.3}$$

Let us assume that $0 < c_1'' < \omega''/2$. Then the torus Λ is divided into four annuli

$$A_1 = \{0 < t'' < c_1''\}, \ A_2 = \{c_1'' < t'' < \omega''/2\} ,$$
$$A_3 = \{\omega''/2 < t'' < c_1'' + \omega''/2\}, \ A_4 = \{c_1'' + \omega''/2 < t'' < \omega''\} .$$

The fixed points of τ_1 are, by (5.22),

$$c_1/2, (c_1 + 1)/2 \in A_1 , \tag{7.4}$$

$$(c_1 + i\omega'')/2, (c_1 + 1 + i\omega'')/2 \in A_3 . \tag{7.5}$$

For the map $z = \pi_1(t)$ (6.3), we get

$$D_j = \pi_1(A_j),\ C_j = \partial D_j,\ 1 \leqslant j \leqslant 4\ . \tag{7.6}$$

Then the z-plane is the disjoint union of $D_1, D_2 = D_4, D_3, C_1$, and C_3. π_1 maps each of A_2 and A_4 biholomorphically onto D_2, which is topologically an annulus with boundary $C_2 = C_1 - C_3$. π_1 also gives twofold branched coverings of A_i onto D_i, $i = 1, 3$, branched at (7.4), (7.5). In particular, D_1 is unbounded, containing $\pi_1(c_1/2) = \infty$ in its interior, and C_1 and C_3 are symmetric with respect to the real axis. $\pi_1(t)$ is real on the two horizontal lines through (7.4) and (7.5). It is also real on the two vertical lines $\{t' = c_1'/2\}$, $\{t' = (c_1' + 1)/2\}$, which intersect ∂A_1 in the points

$$a_1 = \frac{1}{2} c_1',\ a_2 = \frac{1}{2} c_1' + ic_1''\ , \tag{7.7}$$

and

$$b_1 + 1 = \frac{1}{2}(c_1' + 1),\ b_2 = \frac{1}{2}(c_1' + 1) + ic_1''\ . \tag{7.8}$$

$\pi_1(a_1) = \pi_1(a_2) \in C_1 \cap \mathbf{R}$ and $\pi_1(b_1 + 1) = \pi_1(b_2) \in C_1 \cap \mathbf{R}$ are the "vertices" of C_1.

The "annular" domain D_2 has the same conformal type as A_2, which is determined by $\frac{1}{2}\omega'' - c_1''$. This depends on both Λ and τ_1.

Finally we consider the Riemann map, $\zeta = f(z)$, of D_1 onto the right half plane H, which takes $\pi_1(a_1)$ to zero and $\pi_1(b_1 + 1)$ to ∞. We extend $f \circ \pi_1(t)$ to the entire t-plane by reflection in the lines $\{t'' = 0\}$ and $\{t'' = c_1''\}$. This gives a doubly periodic meromorphic function $\hat{\varphi}$ with period module

$$\hat{\Lambda} = \{n_1 \cdot 1 + n_2 \cdot 2c_1''i \mid n_1, n_2 \in \mathbf{Z}\}\ . \tag{7.9}$$

$\hat{\varphi}$ has the representation in terms of the sigma function $\hat{S}$ relative to $\hat{\Lambda}$,

$$\hat{\varphi} = \frac{\hat{S}(t - a_1)\hat{S}(t - a_2)}{\hat{S}(t - b_1)\hat{S}(t - b_2)}\ . \tag{7.10}$$

The invariance $\hat{\varphi} \circ \tau_1 = \hat{\varphi}$ follows as in section 4, since $\tau_1(a_1) = a_2$, $\tau_1(b_1) = b_2$. Since $z = \mathscr{P}(t - c_1/2)$, we have

THEOREM 7.1. *Let Λ be the lattice with periods (5.19), and let $D_1 \subset \mathbf{P}_1$ be the simply connected domain above. The Riemann map, $\zeta = f(z)$, of D_1 onto the right half ζ-plane H is given by*

$$\zeta = \hat{\varphi}(\mathscr{P}^{-1}(z) + c_1/2)\ , \tag{7.11}$$

where $\hat{\varphi}$ *is the sigma quotient (7.10) relative to the lattice (7.9), and* $\mathcal{P}^{-1}(z)$ *is the elliptic integral of the first kind, in Weierstrass normal form, relative to* Λ.

REMARK. We have seen that double valued reflection places a severe restriction on a real algebraic curve in the complex plane. In fact our results should provide the basis for a complete and explicit classification. We have also seen how double valued reflection may be used to explicitly determine Riemann maps. Apparently, all known such examples can be so explained. The result in the above theorem seems to be new. It would be interesting to work out more examples in the genus one case.

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