ELLIPTIC SPACES II

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ABSTRACT. A simply connected finite CW complex X is *elliptic* if the homology of its loop space (coefficients in any field) grows at most polynomially. We show that in all other cases the loop space homology grows at least semi-exponentially, and we exhibit a number of geometrically interesting classes of spaces as elliptic, including: H spaces, homogeneous spaces, Poincaré duality complexes whose mod p cohomology is doubly generated (any p) and Dupin hypersurfaces in S^{n+1}.

1. INTRODUCTION

Let X be a simply connected finite CW complex, with loop space ΩX, and denote by $\mathbf{F}_p$, the prime field of characteristic p, p prime or zero. Our first main result asserts a dichotomy for the size of the loop space homology $H_*(\Omega X; \mathbf{F}_p)$:

THEOREM A. *Let X be a simply connected finite CW complex. For each p (prime or zero) there are exactly two possibilities: either*

(i) *There are constants $C > 0$ and $r \in \mathbf{N}$ such that*

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbf{F}_p) \leqslant Cn^r, \quad n \geqslant 1 ,$$

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or else

(ii) *There are constants* $K > 1$ *and* $N \in \mathbf{N}$ *such that*

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbf{F}_p) \geqslant K^{\sqrt{n}}, \qquad n \geqslant N.$$

In case (i) the loop space homology grows *at most polynomially,* and X is $\mathbf{Z}_{(p)}$-elliptic in the sense of [6]. If (i) holds for all p then X is elliptic. The main theorems of [6] assert that if X is elliptic then X is a Poincaré complex and that $H_*(\Omega X; \mathbf{Z})$ is a finitely generated left noetherian ring.

In case (ii) above the loop space homology grows *at least semi-exponentially*. However, when $p = 0$ [2] or $p \geqslant \dim X$ [8], it can be shown that even the primitive subspace of $H_*(\Omega X; \mathbf{F}_p)$ grows exponentially (implying the same result for $H_*(\Omega X; \mathbf{F}_p)$), and we conjecture that this should hold true for all p.

In the dichotomy of Theorem A, the generic situation is (ii): elliptic spaces are rare within the class of all simply connected finite CW complexes. However a number of geometrically interesting spaces are elliptic, and our second objective in this note is to show that these include the following classes of spaces (provided they are simply connected):

finite H-spaces,

homogeneous spaces,

spaces admitting a fibration $F \to X \to B$ with F, B elliptic,

Poincaré complexes X such that for each p, the algebra $H^*(X; \mathbf{F}_p)$ is generated by two elements,

Dupin hypersurfaces in S^{n+1},

closed manifolds admitting a smooth action by a compact Lie group, with a simply connected codimension one orbit,

connected sums $M \# N$ with the algebras $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ each generated by a single class.

This note is sequel to "Elliptic Spaces" [6]. In particular, it supersedes the preprint "Dupin hypersurfaces are elliptic" referred to in [6].

2. The dichotomy

Consider first any simply connected space X with each $H_i(X; \mathbf{F}_p)$ finite dimensional. Then $G = H_*(\Omega X; \mathbf{F}_p)$ is a graded cocommutative Hopf algebra satisfying $G_0 = \mathbf{F}_p$ and each G_i is finite dimensional. The *depth* of G

is the least integer m such that $\mathrm{Ext}_G^m(\mathbf{F}_p; G) \neq 0$; if $\mathrm{Ext}_G(\mathbf{F}_p; G) \equiv 0$ we say G has *infinite depth.* In [3: Theorem A] it is shown that

$$\operatorname{depth} H_*(\Omega X; \mathbf{F}_p) \leqslant LS \operatorname{cat} X .$$

Thus the depth is finite when X has the weak homotopy type of a finite CW complex.

On the other hand suppose G is any graded cocommutative Hopf algebra with $G_0 = \mathbf{F}_p$ and each G_i finite dimensional. We call G *elliptic* [7] if G is a finitely generated nilpotent Hopf algebra. According to [4; Theorem A] this is equivalent to the condition:

$$\operatorname{depth} G < \infty \qquad \textit{and} \qquad \sum_{i=0}^{n} \dim G_i \leqslant Cn^r \,(\textit{fixed } C, r, \textit{all } n) .$$

In view of these remarks, Theorem A follows from

THEOREM 2.1. *Let G be a cocommutative Hopf algebra of finite depth such that $G_0 = \mathbf{F}_p$ and each G_i is finite dimensional. Then there are exactly two possibilities:*

(1) *G is elliptic, and for some $r \in \mathbf{N}$ there are positive constants C_1, C_2 such that*

$$C_1 n^r \leqslant \sum_{i=0}^{n} \dim G_i \leqslant C_2 n^r , \qquad n \geqslant 1 ;$$

(2) *For some constants $K > 1, N \in \mathbf{N}$*

$$\sum_{i=0}^{n} \dim G_i \geqslant K^{\sqrt{n}} , \qquad n \geqslant N .$$

Proof. Consider the formal power series $G(z) = \sum_{i=0}^{\infty} \dim G_i z^i$, and for two formal power series $f = \sum_{i=0}^{\infty} a_i z^i$ and $g = \sum_{i=0}^{\infty} b_i z^i$ write $f \underset{c}{\ll} g$ if

$$\sum_{i=0}^{n} a_i \leqslant \sum_{i=0}^{n} b_i , \qquad \text{all } n . \tag{2.1}$$

We shall first show that there are exactly two possibilities:

(2.2) *For some $r \in \mathbf{N}$ there are positive constants C_1, C_2 such that*

$$C_1 n^r \leqslant \sum_{i=0}^{n} \dim G_i \leqslant C_2 n^r , \qquad n \geqslant 1 ;$$

(2.3) *For some* $k \in \mathbf{N}$.

$$G(z) \underset{c}{\geqslant} \prod_{i=1}^{\infty} [1 + (z^k)^i] \, .$$

Indeed, suppose $\sum_{i=0}^{n} \dim G_i \leqslant C_2 n^r$ for all n, some C_2 and r. Then by [4; Theorem B], G is elliptic and hence [7; Prop. 3.6] the formal power series $G(z)$ has the form

$$G(z) = \frac{\prod_{j=1}^{s} (1 + z^{k_j} + \cdots + z^{(n_j - 1)k_j})}{\prod_{i=1}^{r} (1 - z^{l_i})} \, .$$

It follows at once that (2.2) is satisfied.

Conversely, we assume there is no C, r for which $\sum_{i=0}^{n} \dim G_i \leqslant Cn^r$, all n, and prove (2.3). Let $x_1, x_2, \ldots$ be a sequence of generators of the algebra G with $\deg x_1 \leqslant \deg x_2 \leqslant \cdots$. The subalgebra $G(i)$ generated by $x_1, \ldots, x_i$ is then a sub Hopf algebra. Now according to [4; Prop. 3.1] there is some q such that $G(i)$ has finite depth, $i \geqslant q$. Moreover by [7; Prop. 3.5] $G(l)$ is not elliptic for some $l \geqslant q$. Set $H = G(l)$; it is a finitely generated non-elliptic Hopf algebra of finite depth, and $\dim G_i \geqslant \dim H_i$.

Next, let R be the sum of the solvable normal sub Hopf algebras of H. Then [3; Theorem C] R is elliptic. Hence [7; Prop. 3.1] and [3; Prop. 3.1] the quotient Hopf algebra $H /\!/ R$ has finite depth, but [7; Prop. 3.3] $H /\!/ R$ is not elliptic. Clearly, however, $H /\!/ R$ is finitely generated and has no central primitive elements. Now by [4; Prop. 3] there is an integer n_0 and an infinite sequence of non zero primitive elements $y_i \in H /\!/ R$ such that for all i, $\deg y_i \leqslant \deg y_{i+1} \leqslant \deg y_i + n_0$. A linear embedding

$$\bigotimes_{i=1}^{\infty} \mathbf{F}_p[y_i]/y_i^2 \to H /\!/ R$$

is then defined by $y_1^{\varepsilon_1} \otimes \cdots \otimes y_m^{\varepsilon_m} \to y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m}$, and so

$$\prod_{i=1}^{\infty} (1 + z^{\deg y_i}) \underset{c}{\leqslant} (H /\!/ R)(z) \underset{c}{\leqslant} H(z) \underset{c}{\leqslant} G(z) \, .$$

Since $\deg y_{i+1} \leqslant i n_0 + \deg y_1$ it is sufficient to take $k = \max(\deg y_1, n_0)$ to achieve (2.3).

It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series $h(z)$ it will also hold for $h(z^k)$, at the cost of replacing K by $K^{\frac{1}{2k}}$. By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1 + z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10]. $\square$

COROLLARY OF PROOF. *If G satisfies the hypotheses of Theorem 2.1 (2) then for some $k \in \mathbf{N}$,*

$$G(z) \underset{c}{\geqslant} \prod_{i=1}^{\infty} [1 + (z^k)^i] . \quad \square$$

3. ELLIPTIC SPACES

In this section we establish the ellipticity of the spaces listed in the introduction.

3.1. Finite simply connected H-spaces, X.

Because X is an H-space, $H_*(\Omega X; \mathbf{F}_p)$ is commutative, all p. Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence X is elliptic.

3.2. Simply connected homogeneous spaces, $G // H$.

We may suppose that G is simply connected, and hence elliptic by § 3. The fibration $G \to G/H \to BH$ loops to the fibration $\Omega G \to \Omega(G/H) \to H$ in which $\pi_1(H)$ acts trivially in $H_*(\Omega G; \mathbf{F}_p)$ [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for $H_*(\Omega(G/H); \mathbf{F}_p)$ from the same property for $H_*(\Omega G; \mathbf{F}_p)$.

3.3. Fibrations $F \to X \to B$ with F, B elliptic.

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that $H_*(X; \mathbf{Z})$ is concentrated in finitely many degrees, and finitely generated in each. Hence X has the weak homotopy type of a finite CW complex. Loop the fibration $F \to X \to B$ and use the fact that $H_*(\Omega F; \mathbf{F}_p)$ and $H_*(\Omega B; \mathbf{F}_p)$ grow polynomially to deduce the same property for $H_*(\Omega X; \mathbf{F}_p)$.

3.4. Simply connected Poincaré complexes X with $H^*(X; \mathbf{F}_p)$ at most doubly generated.

Suppose $p \neq 2$ and $H = H^*(X; \mathbf{F}_p)$ contains an element of odd degree. Then it has an odd generator α. Using Poincaré duality it is easy to see that there are only three possibilities for the algebra H:

$$H = \Lambda\alpha \quad \text{or} \quad \Lambda\alpha \otimes \Lambda\beta \quad \text{or} \quad \Lambda\alpha \otimes \mathbf{F}_p[\beta]/\beta^k .$$

In each case a simple, classical computation [11] produces $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ and shows that it grows polynomially. Since the Eilenberg-Moore spectral sequence converges from $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ to $H^*(\Omega X; \mathbf{F}_p)$, $H^*(\Omega X; \mathbf{F}_p)$ also has this property.

In all other cases ($p = 2$ or H concentrated in even degrees) H is a commutative local ring in the classic sense. Because H satisfies Poincaré duality it is a Gorenstein ring. Now a theorem of Wiebe [12; Korollar p. 268] asserts (because H has at most two generators) that H is a polynomial algebra divided by a regular sequence. It is thus easy (and classical [11]) to compute $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$, and deduce that it grows polynomially. Hence so does $H_*(\Omega X; \mathbf{F}_p)$.

3.5. Simply connected Dupin hypersurfaces E in S^{n+1}.

In [9; Table 2.1] are listed the possibilities for $H_*(E; \mathbf{Z})$. We divide these into three cases, using the notation of [9].

Case (a): E has the same integral homology as S^k or as $S^k \times S^l$.

In this case Poincaré duality shows that E has the same integral cohomology ring as S^k or as $S^k \times S^l$, and we can apply 3.4.

Case (b): E has the rational homotopy type of $A_3(2)$, $A_3(4)$, $A_3(8)$, $A_4(2)$ or $A_6(2)$.

In these cases the calculations of [9; §6] show explicitly that the ring $H^*(E; \mathbf{Z})$ is torsion free and generated by two elements. Thus each $H^*(E; \mathbf{F}_p)$ is doubly generated, and we can apply Wiebe's result as in 3.4.

Case (c): E has the integral homology of $S^k \times S^l \times S^{k+l}$, with $k < l$.

We need, in this case, to recall from [9; §2] that there are linear sphere bundles

$$S^k \to E \xrightarrow{\pi_0} B \quad \text{and} \quad S^l \to E \xrightarrow{\pi_1} B_1$$

with B_0, B_1 simply connected focal submanifolds of S^{n+1}. Moreover if D_0, D_1 denote the corresponding disk bundles with boundary E then $S^{n+1} = D_0 \underset{E}{\cup} D_1$.

Fix $p \geqslant 0$ and consider the Serre spectral sequence for the fibration $S^k \to E \to B_0$ with coefficients in $\mathbf{F}_p$. If this fails to collapse then $H^k(\pi_0): H^k(B_0; \mathbf{F}_p) \to H^k(E; \mathbf{F}_p)$ is surjective. Since $l > k$ it is always true that $H^k(\pi_1)$ is surjective. Choose classes $\alpha \in H^k(B_0; \mathbf{F}_p)$, $\beta \in H^k(B_1; \mathbf{F}_p)$ mapping to the same non-zero class in $H^k(E; \mathbf{F}_p)$. The Mayer-Vietoris sequence for the decomposition $S^{n+1} = D_0 \underset{E}{\cup} D_1$ then gives a class $\gamma \in H^k(S^{n+1}; \mathbf{F}_p)$ restricting to α and β, which is absurd.

Thus the spectral sequence for $S^k \to E \to B_0$ collapses and so $H_*(B_0; \mathbf{F}_p) \cong H_*(S^l \times S^{l+k}; \mathbf{F}_p)$. Using Poincaré duality for B_0 we see that $H^*(B_0; \mathbf{F}_p)$ and $H^*(S^l \times S^{l+k}; \mathbf{F}_p)$ are isomorphic as graded algebras. Thus B_0 is elliptic by 3.4 and E is elliptic by 3.3.

3.6. Simply connected closed manifolds M with a smooth action by a compact Lie group G, having a simply connected codimension one orbit.

Here we may assume G is connected. Let the orbit be G/K, and convert the inclusion of G/K into a fibration $F \to G/K \to M$. From [9; Table 1.5] we see that for any p, $\dim H_i(F; \mathbf{F}_p) \leqslant 2$, all i. Thus applying the Serre spectral sequence to the fibration $\Omega(G/K) \to \Omega M \to F$ and using 3.1 for G/K we see that $H_*(\Omega M; \mathbf{F}_p)$ grows polynomially.

3.7. Simply connected manifolds $M \# N$ with each of the rings $H^*(M; \mathbf{Z})$, $H^*(N; \mathbf{Z})$ generated by a single class.

By Van Kampen's theorem both M and N are simply connected, and so their fundamental cohomology classes are not torsion. Since each ring is monogenic, $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ are torsion free. Thus $H^*(M; \mathbf{F}_p)$ and $H^*(N; \mathbf{F}_p)$ are also monogenic, and so $H^*(M \# N; \mathbf{F}_p)$ is doubly generated. Now apply 3.4.

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