# §4. Automorphisms of the root System $nD_4$ and perfect isometries

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*Proof.* Note that every perfect isometry $\sigma$ of $\mathcal{H}$ extends naturally to a perfect isometry of $\mathcal{H}^*$, inducing a perfect $\mathbf{F}_2$-isomorphism $\eta(\sigma)$ of $\mathcal{H}^*/\mathcal{H}$, $\eta$ denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it corresponds to multiplication by $\omega$, where $\omega$ denotes a primitive element of $\mathbf{F}_4$ over $\mathbf{F}_2$.*

*Proof.* An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of $\mathbf{F}_4$, corresponds to multiplication by $\omega$, $\omega$ being as above.

3.6. PROPOSITION. *Let $L$ be a $\mathbf{Z}$-lattice such that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*^n}$. If $L$ is an $\mathcal{H}$-lattice, then $L$ has a perfect isometry, which corresponds to multiplication by $\omega$, on the quotient $\mathcal{H}^{*^n}/\mathcal{H}^n$.*

*Proof.* Multiplication by $\xi$ is a perfect isometry of $\mathcal{H}^n$ which extends naturally to a perfect isometry of $\mathcal{H}^{*^n}$. Clearly the induced map on the quotient $\mathcal{H}^{*^n}/\mathcal{H}^n$ is multiplication by $\omega$. Since $L$ is an $\mathcal{H}$-module, it preserves $L$ as well.

In particular,

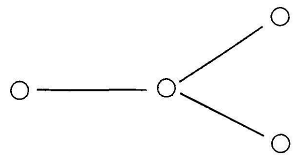
3.7. COROLLARY. *Every $\mathcal{H}$-lattice $(L, Tr \circ h)$ of type $n\mathrm{D}_4$ has a perfect isometry.*

It is but natural to ask whether every $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ which has a perfect isometry necessarily admits the structure of an $\mathcal{H}$-lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system $n\mathrm{D}_4$.

## §4. AUTOMORPHISMS OF THE ROOT SYSTEM $n\mathrm{D}_4$ AND PERFECT ISOMETRIES

For any root system R, let $\mathcal{W}(\mathrm{R})$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathcal{W}(\mathrm{R})$ is a normal subgroup of $Aut\,\mathrm{R}$, which preserves every $\mathbf{Z}$-lattice $L$ such that $\mathbf{Z}\mathrm{R} \subseteq L \subseteq \mathbf{Z}\mathrm{R}^{\#}$. We thus get a natural map $\eta: Aut\,\mathrm{R}/\mathcal{W}(\mathrm{R}) \to Aut_{\mathbf{Z}}(\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R})$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.

An element $\sigma$ in $Aut(\mathrm{R})/\mathscr{W}(\mathrm{R})$ preserves $L$ if and only if $\eta(\sigma)$ preserves the corresponding subgroup $\eta(L)$ of $\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R}$. If $\mathrm{R} = \mathrm{D}_4$, $Aut\,\mathrm{R} = \mathscr{W}(\mathrm{R}) \underset{s}{\ltimes} S_3$, where, $\underset{s}{\ltimes}$ denotes the semi direct product and $S_3$ is the automorphism group of the associated Dynkin diagram:



Consequently, for $\mathrm{R} = n\mathrm{D}_4$, $Aut\,\mathrm{R}/\mathscr{W}(\mathrm{R}) \simeq S_3^n \underset{s}{\ltimes} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\ltimes} S_n$. Thus the elements of $Aut\mathrm{R}/\mathscr{W}(\mathrm{R})$ are "monomial matrices" where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. It acts naturally on $(\mathbf{Z}\mathrm{D}_4^{\#})^n/\mathbf{Z}\mathrm{D}_4^n$. In view of the identification of $\mathbf{Z}\mathrm{D}_4^{\#}/\mathbf{Z}\mathrm{D}_4 \simeq \mathscr{H}^*/\mathscr{H}$, we have the following proposition.

4.1. PROPOSITION.

(a) $Aut(\mathscr{H}^n)/\mathscr{W}(\mathscr{H}^n) \simeq S_3^n \underset{s}{\ltimes} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\ltimes} S_n$.

(b) *If $U$ denotes the group of units of $\mathscr{H}$, then $U$ is a subgroup of $Aut\,\mathscr{H}$ and $U/(\mathscr{W}(\mathscr{H}) \cap U) \simeq \{1, \omega, \omega^2\}$, where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.*

(c) *The conjugation in $\mathscr{H}$ belongs to the Weyl group $\mathscr{W}(\mathscr{H})$.*

*Proof.* (a) This statement is an immediate consequence of the identification $\mathbf{Z}\mathrm{D}_4 \simeq \mathscr{H}$.

(b) By (a), $Aut\,\mathscr{H}/\mathscr{W}(\mathscr{H}) \simeq S_3 \simeq GL_2(\mathbf{F}_2)$. Since $\eta(U) = \{1, \omega, \omega^2\}$, (b) follows.

(c) The conjugation in $\mathscr{H}$ is a product of reflections defined by $i, j$ and $k$.

We now consider the perfect isomorphisms of $(\mathscr{H}^{*n})/\mathscr{H}^n$ arising out of $Aut(\mathscr{H}^n)/\mathscr{W}(\mathscr{H}^n)$. We begin by fixing the following notation:

Let $V = \mathbf{F}_4^n = X_1 \perp X_2 \perp \dots X_n$ with respect to the standard hermitian form on $V$, where $X_i \simeq \mathbf{F}_4 = \mathbf{F}_2 \oplus \mathbf{F}_2 = \{0, 1, \omega, \omega^2\}$. Let $G$ denote the group of all $n \times n$ monomial matrices with entries in $M_2(\mathbf{F}_2)$, where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. Note that every element of $G$ can be uniquely expressed as $\alpha \,.\, \tau$, where $\alpha$ is the diagonal matrix $\mathrm{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, with $\alpha_i$ in $GL_2(\mathbf{F}_2)$ and $\tau$ is an $n \times n$ permutation matrix. We have, 4.2. LEMMA. *Let* $\sigma$ *belonging to* $G$ *be perfect and let* $X = X_i$ *for some* $i$. *Let* $m$ *be the smallest positive integer for which* $\sigma^m$ *maps* $X$ *onto itself. Then* $\sigma^m/X$ *is perfect.*

*Proof.* The idea of the proof is similar to ([K], Prop. 2). We show that $(1-\sigma^m)/X$ is surjective. Let $M = \sum_{0 \leqslant i \leqslant m-1} \sigma^i(X)$. Then $\sigma$ leaves $M$ invariant. Therefore $\sigma$ is a perfect isomorphism of $M$. Hence $(1-\sigma)/M$: $M \to M$ is surjective. Let $x$ be an element of $X$. Since, $(x, 0, \ldots, 0)$ belongs to $M$, there exists an element $y$ in $M$ such that $(1-\sigma)(y) = (x, 0, \ldots, 0)$. Let $y = (y_0, y_1, \ldots, y_{m-1})$, where $y_i$ belongs to $\sigma^i(X)$. Then,

$$(1-\sigma)(y) = (y_0 - \sigma(y_{m-1}), \quad y_1 - \sigma(y_0), \ldots, y_{m-1} - \sigma(y_{m-2})) .$$

Hence, $y_0 - \sigma(y_{m-1}) = x, y_1 = \sigma(y_0), \ldots, y_{m-1} = \sigma(y_{m-2})$. Further, $\sigma(y_{m-1}) = \sigma^2(y_{m-2}) = \ldots = \sigma^m(y_0)$. Thus $(1-\sigma^m)(y_0) = x$. This implies that $(1-\sigma^m)/X$ is surjective.

4.3. COROLLARY. *Let* $\sigma$ *be an element of* $G$ *which is perfect. Suppose that* $\sigma = \alpha . \tau$, *where* $\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$, $\alpha_i \in GL_2(\mathbf{F}_2)$, $\tau = \tau_1 . \tau_2 \ldots \tau_r$, *and* $\tau_i$ *are disjoint cyclic permutations of length* $n_i$. *Let* $T_i$ *denote the set of indices belonging to the permutation* $\tau_i$. *Then* $(\sigma)^{n_i}/X_j$ *is perfect for every* $j$ *belonging to* $T_i$.

*Proof.* Note that for every $j$ belonging to $T_i$, $n_i$ is the smallest positive integer such that $(\sigma)^{n_i}$ maps $X_j$ onto itself.

4.4. COROLLARY. If $\sigma$ *is as above, then* $(\sigma)^{n_i}/X_j$ *corresponds to multiplication by* $\omega$ *or* $\omega^2$, *for every* $j$ *belonging to* $T_i$.

*Proof.* Follows from Corollary 4.3, and Lemma 3.5.

4.5. COROLLARY. *If* $\sigma$ *is as above, and* $X^{(i)} = \sum_{j \in T_i} X_j$, *then* $(\sigma)^{n_i}/X^{(i)}$ *is the matrix* $\operatorname{diag}(\alpha_1, \ldots, \alpha_j, \ldots \alpha_{n_i})$, *where* $\alpha_j$ *belongs to* $\{\omega, \omega^2\}$.

*Proof.* Clear from Corollary 4.4.

4.6. PROPOSITION. *Let* $\sigma$ *be an element of* $G$ *which is perfect and let* $\sigma = \alpha . \tau$, *where* $\alpha$ *and* $\tau$ *are as in Corollary 4.4. Then there exists an integer* $l \geqslant 1$, *such that* $\sigma^l$ *is perfect and* $\sigma^l = \beta . \tau'$, *where* $\beta$ *is the matrix* $\operatorname{diag}(\beta_1, \ldots, \beta_j, \ldots, \beta_n)$, *with* $\beta_j$ *in* $GL_2(\mathbf{F}_2)$ *and* $\tau'$ *is a product of disjoint cyclic permutations* $\tau_i$ *of length* $3^{k_i}$.

*Proof.* Let $\tau = \tau_1 . \tau_2 \dots \tau_r$, where $\tau_i$ are disjoint cyclic permutations of length $n_i = 3^{k_i} . l_i$, with $(3, l_i) = 1$. Let $l$ denote the least common multiple of the $l_i$. We show that $\sigma^l$ is perfect. By Corollary 4.5, $\sigma^{n_i}/X_j$ is multiplication by $\omega$ or $\omega^2$ for every $j$ belonging to $T_i$. This implies that $(\sigma)^{n_i l/l_i}/X_j$ corresponds to multiplication by $\omega$ or $\omega^2$ for every such $j$, since $(l/l_i, 3) = 1$ and $\omega$ is an element of order 3. Hence, $(\sigma^l)^{3^{k_i}}/X^{(i)}$ is the matrix $\operatorname{diag}(\alpha_1, \dots, \alpha_j, \dots \alpha_{n_i})$ where $\alpha_j$ belongs to $\{\omega, \omega^2\}$. Clearly this implies that $\sigma^l/X^{(i)}$ has no nontrivial fixed point. Since $T_i$ are disjoint, it follows that $\sigma^l$ has no nontrivial fixed point and hence $\sigma^l$ is perfect. Obviously $\sigma^l$ has the required property and the proposition follows.

Now, let $M$ be an $\mathbf{F}_2$-linear subspace of $V$, which is invariant under a perfect isomorphism $\sigma$ belonging to $G$. By the previous proposition, we can assume, by replacing $\sigma$ by $\sigma^m$, that $M$ is invariant under $\sigma = \alpha . \tau$, where $\alpha$ is as in Corollary 4.4 and $\tau = \tau_1 . \tau_2 \dots \tau_r$, $\tau_i$ being cyclic permutations of length $3^{k_i}$.

4.7. PROPOSITION. *If $M$ is an $\mathbf{F}_2$-linear subspace of $V$ which has a perfect isomorphism $\sigma$ belonging to $G$, then $M$ is invariant under the action of a diagonal matrix,* $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ *where each $\alpha_i$ belongs to* $\{\omega, \omega^2\}$.

*Proof.* By replacing $\sigma$ by a suitable power we may assume that

$$\sigma = \operatorname{diag}(\beta_1, \dots, \beta_i, \dots, \beta_n)\, \tau_1 \tau_2 \dots \tau_r$$

where $\beta_i$ belongs to $GL_2(\mathbf{F}_2)$ for every $i$ and $\tau_i$ are disjoint cyclic permutations of length $3^{k_i}$. Further, since disjoint cycles commute we may assume that the length of $\tau_i$ is $3^k$ for $1 \leqslant i \leqslant s$ and the length of $\tau_i$ is less than $3^k$ for $s < i \leqslant r$. Let $T = \{i \in \{1, 2, \dots, n\} \mid i$ occurs in the permutation $\tau_1 \tau_2 \dots \tau_s\}$. Let $M_1 = M \cap \sum_{i \in T} X_i$ and $N_1 = M \cap \sum_{i \notin T} X_i$. We claim that $M = M_1 \oplus N_1$ and that $M_1$ is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, where each $\alpha_i$ belongs to $\{\omega, \omega^2\}$. Let $(x, y) \in M$, where $x \in \perp_{i \in T} X_i$, $y \in \perp_{i \notin T} X_i$. Since

$$\sigma^{3^k} = \operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)\ ,$$

where $\alpha_i$ belongs to $\{\omega, \omega^2\}$ for $i \in T$ and $\alpha_i = 1$ for $i \notin T$, it follows that, $(x, y) + \sigma^{3^k}(x, y) + (\sigma^{3^k})^2(x, y) = (0, y)$ belongs to $M$. Hence $(x, 0)$ belongs to $M$ as well. Thus $M = M_1 \oplus N_1$. Clearly $M_1$ is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, $\alpha_i$ being in $\{\omega, \omega^2\}$. Since $\sigma/N_1$ is perfect, by repeating the above argument we obtain a similar decomposition of $N_1 : N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each $M_j$ is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots \alpha_n)$, $\alpha_i$ being in $\{\omega, \omega^2\}$.

## §5. Main Theorem and examples

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. Proposition. *Let $L$ be a unimodular* **Z**-*lattice of type $n\mathrm{D}_4$ such that $\mathscr{H}^n \subset L \subset \mathscr{H}^{*n}$. If $L$ admits a perfect isometry, then there exists an isometry $\delta = \operatorname{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on $\mathscr{H}^{*n}$, where $\delta_i$ is the isometry on $\mathscr{H}^*$ given by left multiplication by $\xi$ or right multiplication by $\bar{\xi}$ such that $L$ is invariant under $\delta$.*

*Proof.* Let $\sigma$ be a perfect isometry of $(L, Tr \circ h)$. Then $\sigma$ induces an automorphism of $\mathscr{H}^n$ and extends naturally to a perfect isometry of $\mathscr{H}^{*n}$. In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of $\mathbf{F}_4^n$, leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with $\alpha_i$ in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under $\alpha$. Let $\delta_i$ denote left multiplication on $\mathscr{H}^*$ by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \operatorname{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since $\delta$ induces an isometry of $\mathscr{H}^{*n}$ which fixes $\mathscr{H}^n$ and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that $\delta$ leaves $L$ invariant.

5.2. Theorem. *Let $(L, S)$ be an unimodular* **Z**-*lattice of type $n\mathrm{D}_4$. Then, $L$ has a perfect isometry if and only if there exists an $\mathscr{H}$-lattice $(L', S')$ such that $L \simeq L'$.*

*Proof.* Clearly every $\mathscr{H}$-lattice admits a perfect isometry (3.2). Conversely let $(L, S)$ be a **Z**-lattice of type $n\mathrm{D}_4$, which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset $T$ of $\{1, 2, \dots, n\}$ such that $L$ is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where $\delta_i$ is left multiplication by $\xi$ for $i \in T$ and $\delta_i$ is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f : \mathscr{H}^n \to \mathscr{H}^n$ be defined by $f = \operatorname{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \mathrm{id}$ for $i \in T$ and $f_i =$ the involution on $\mathscr{H}$ for $i \notin T$. Then it is easy to check that $f$ is an isometry of $(L, Tr \circ h)$ onto $(L', S')$ where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i) .$$