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PROPOSITION 9. *Let* $(F, | |)$ *be a field with a non-archimedean absolute value, and suppose the residue field is contained in the algebraically closed field* R. *Define* K *and* L *as the Mal'cev-Neumann fields with value group* $\mathbf{R}$ *and residue field* R. *(Define the p-adic Mal'cev-Neumann field* L *only if* $\operatorname{char} R > 0$.*) The valuations on* K *and* L *induce corresponding absolute values. Then there exists an absolute value-preserving embedding of fields* $\phi: F \to K$ *or* $\phi: F \to L$, *depending on if the restriction of* $| |$ *to the minimal subfield of* F *is the trivial absolute value (on* $\mathbf{Q}$ *or* $\mathbf{F}_p$*) or the p-adic absolute value on* $\mathbf{Q}$.

Similarly, Proposition 8 above gives a glueing proposition for non-archimedean absolute values. In fact, this result holds for archimedean absolute values as well, in light of Ostrowski's theorem.

7. EXAMPLE: THE MAXIMALLY COMPLETE IMMEDIATE EXTENSION OF $\bar{\mathbf{Q}}_p$

For this section, (L, v) will denote the p-adic Mal'cev-Neumann field having value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$. We have a natural embedding of $\mathbf{Q}_p$ into L. By Corollary 4, L is algebraically closed, so this embedding extends to an embedding of $\bar{\mathbf{Q}}_p$ into L (which is unique up to automorphisms of $\bar{\mathbf{Q}}_p$ over $\mathbf{Q}_p$.) In fact this embedding is continuous, since there is a unique valuation on $\bar{\mathbf{Q}}_p$ extending the p-adic valuation on $\mathbf{Q}_p$. Since $\bar{\mathbf{Q}}_p$ has value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$, L is an immediate extension of $\bar{\mathbf{Q}}_p$. By Corollary 6, L is in fact the unique maximally complete immediate extension of $\bar{\mathbf{Q}}_p$. Also, any valued field (F, w) of characteristic 0 satisfying

(1) The restriction of w to $\mathbf{Q}$ is the p-adic valuation;

(2) The value group is contained in $\mathbf{Q}$;

(3) The residue field is contained in $\bar{\mathbf{F}}_p$;

can be embedded in L, by Corollary 5. For example, the completion $\mathbf{C}_p$ of $\bar{\mathbf{Q}}_p$ can be embedded in L. (This could also be proved by noting that L is complete by Corollary 4.)

We will always use as the set S of representatives for $\bar{\mathbf{F}}_p$ the primitive k^{th} roots of 1, for all k not divisible by p, and 0. Then the elements of L have the form $\sum_g \alpha_g p^g$ for some primitive k^{th} roots α_g of 1, where the exponents form a well-ordered subset of $\mathbf{Q}$. In particular, the elements of $\bar{\mathbf{Q}}_p$ can be expressed in this form. This was first discovered by Lampert [9].

Example: (similar to those in [9]) Let p be an odd prime. The p^{th} roots of $1 - p$ in $\bar{\mathbf{Q}}_p$ have the expansion

$$1 - p^{1/p} + p^{1/p + 1/p^2} - p^{1/p + 1/p^2 + 1/p^3} + \cdots$$
$$+ \zeta p^{1/(p-1)} + \text{(higher order terms)} ,$$

where ζ is any one of the p solutions to $\zeta^p = -\zeta$ in $\bar{\mathbf{Q}}_p$.

PROPOSITION 10. *The fields* L *and* $\mathbf{Q}_p$ *have cardinality* $2^{\aleph_0}$ *(and hence so do all intermediate fields).*

Proof. Each series in L defines a distinct function $\mathbf{Q} \to \bar{\mathbf{F}}_p$ by sending q to the residue class of the coefficient of p^q. The number of such functions is $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, so $|L| \leqslant 2^{\aleph_0}$. On the other hand, as is well known, $|\mathbf{Q}_p| = 2^{\aleph_0}$ already, so the result follows. $\square$

Since L and $\mathbf{C}_p$ are both complete algebraically closed fields of cardinality $2^{\aleph_0}$, it is natural to ask if $L = \mathbf{C}_p$. That L strictly contains $\mathbf{C}_p$ follows from Lampert's remark that the support of the series of an element of $\bar{\mathbf{Q}}_p$ is contained in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N, and that the residue classes of the coefficients in the series lie in $\mathbf{F}_q$ for some q. (For example, $p^{-1} + p^{-1/2} + p^{-1/3} + \cdots$ is an element of L which cannot be approached by elements of $\bar{\mathbf{Q}}_p$.) In fact, we can show that the set of series with these properties forms an algebraically closed field, using the following lemma, which is of interest in its own right, and which we can apply also toward the computation of the algebraic closure of Laurent series fields.

LEMMA 5. *Suppose* E *is an algebraically closed field, and* $S \subseteq \operatorname{Aut}(E)$. *Let* F *be the set of elements* $e \in E$ *whose orbit* $\{\sigma(e) \mid \sigma \in S\}$ *under* S *is finite. Then* F *is an algebraically closed subfield of* E.

Proof. Let $\operatorname{Orb}(x)$ denote the orbit of x under S. If $x, y \in F$, then $\operatorname{Orb}(x+y) \subseteq \operatorname{Orb}(x) + \operatorname{Orb}(y)$ which is finite, so $x + y \in F$. Similar considerations complete the proof that F is a subfield.

Given $p(x) \in F[x]$, let c be a zero of p in E. Then the orbit of $p(x)$ under S is finite (since each coefficient has finite orbit), and $\operatorname{Orb}(c)$ consists of zeros of polynomials in the orbit of $p(x)$ (to be specific, $\sigma(c)$ is a zero of $\sigma(p)$), so $c \in F$. Hence F is algebraically closed. $\square$

The characteristic p case of the following corollary was proved by Rayner [16] using a different method.

COROLLARY 7. *If* k *is an algebraically closed field of characteristic* 0, *then* $k((t)) = \cup_{n=1}^{\infty} k((t^{1/n}))$. *If* k *is an algebraically closed field of characteristic* p, *then the set of series in* $k((\mathbf{Q}))$ *with support in* $\frac{1}{N}\mathbf{Z}[1/p]$ *for some* N *(depending on the series) is an algebraically closed field containing* $k((t))$.

Proof. If ζ is a homomorphism from $\mathbf{Q}/\mathbf{Z}$ to the group of all roots of unity in k, then we get an automorphism of the algebraically closed Mal'cev-Neumann ring $k((\mathbf{Q}))$ by mapping $\sum_{q \in \mathbf{Q}} \alpha_q t^q$ to $\sum_{q \in \mathbf{Q}} \zeta(q) \alpha_q t^q$. Let $E = k((\mathbf{Q}))$ and let S be the set of all such automorphisms. Then the lemma shows that the set F of elements of E with finite orbit under S is an algebraically closed field. If char $k = 0$, $F = \cup_{n=1}^{\infty} k((t^{1/n}))$, and the desired result follows easily. If char $k = p$, F is the set of series in $k((\mathbf{Q}))$ with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N (since ζ is necessarily trivial on $\mathbf{Z}[1/p]/\mathbf{Z}$). □

COROLLARY 8. *The set of series in* L *with support in* $\frac{1}{N}\mathbf{Z}[1/p]$ *for some* N *such that the residue classes of the coefficients lie in* $\mathbf{F}_q$ *for some* q *forms an algebraically closed field which contains* $\mathbf{Q}_p$, *hence also* $\bar{\mathbf{Q}}_p$.

Proof. If μ denotes the group of all k^{th} roots of 1 for all k relatively prime to p, and $\zeta: \mathbf{Q}/\mathbf{Z} \to \mu$ is any group homomorphism, then we get an automorphism of $A((\mathbf{Q}))$ (using the notation of Section 4) by sending $\sum_{g \in \mathbf{Q}} \alpha_g t^g$ to $\sum_{g \in \mathbf{Q}} \zeta(g) \alpha_g t^g$. This maps the ideal N into itself, so it induces an automorphism of L. We also get automorphisms of L coming functorially from the automorphisms of $\bar{\mathbf{F}}_p$.

Let $E = L$, and let S be the set of both types of automorphisms. Then the elements of L with finite orbit under the first type of automorphisms are those with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N, and the elements with finite orbit under the second type of automorphisms are those such that the residue classes of the coefficients lie in $\mathbf{F}_q$ for some q. Hence the result follows from the lemma. (Obviously this field contains $\mathbf{Q}_p$.) □

There are many automorphisms of L besides those used in the previous proof. In fact, L has an enormous number of continuous automorphisms even over $\mathbf{C}_p$.

Proposition 11. *Given* $\mu \in L/\mathbf{C}_p$, *let* $r = \sup_{e \in \mathbf{C}_p} v(\mu - e) \in \mathbf{R}$. *Then for any* $\mu' \in L$ *such that* $v(\mu - \mu') \geqslant r$, *there exists a continuous automorphism of* L *over* $\mathbf{C}_p$ *taking* μ *to* μ'.

Proof. We will extend the inclusion $\mathbf{C}_p \to L$ to an embedding $\mathbf{C}_p(\mu) \to L$ using the proof of Theorem 2 (instead of taking the obvious inclusion). There is no best approximation to μ in $\mathbf{C}_p$, since given any approximation, we can find a better one by subtracting the leading term of the series of the difference. So we are in Case 2 of the proof of Theorem 2, and it follows that we may embed $\mathbf{C}_p(\mu)$ in L by sending μ to any solution $\mu' \in L$ of the inequalities $v(x - e_\sigma) \geqslant g_\sigma$, where e_σ ranges over all elements of $\mathbf{C}_p$ and $g_\sigma = v(\mu - e_\sigma)$. These are satisfied if $v(\mu - \mu') \geqslant r$, by the triangle inequality. Finally, extend this embedding $\mathbf{C}_p(\mu) \to L$ to a continuous endomorphism $L \to L$ using Theorem 2. This endomorphism is an automorphism by Proposition 7. □

Lampert proved that $\mathbf{C}_p$ has transcendence degree $2^{\aleph_0}$ over the completion $\mathbf{C}_p^{\text{unram}}$ of the maximal unramified extension $\mathbf{Q}_p^{\text{unram}}$ of $\mathbf{Q}_p$, and that $\mathbf{C}_p^{\text{unram}}$ has transcendence degree $2^{\aleph_0}$ over $\mathbf{Q}_p$. We now extend this chain of results by calculating the transcendence degree of L over $\mathbf{C}_p$, using the following generalization of a proposition of Lampert's.

Proposition 12. *If* V *is a sub-*$\mathbf{Q}$*-vector space of* $\mathbf{R}$ *containing* $\mathbf{Q}$, *then the set of elements in* L *of which all the accumulation values of the exponents are in* V *form a complete algebraically closed field.*

Proof. The proof is exactly the same as Lampert's proof for the special case $V = \mathbf{Q}$ [9]. □

Corollary 9. L *has transcendence degree* $2^{\aleph_0}$ *over* $\mathbf{C}_p$.

Proof. Let B be a basis for $\mathbf{R}$ as a vector space over $\mathbf{Q}$, with $1 \in B$. For each $b \in B, b \neq 1$, pick a strictly increasing sequence $q_1, q_2, \ldots$ in $\mathbf{Q}$ with limit b, and define $z_b = p^{q_1} + p^{q_2} + \cdots \in L$. Let K_b be the field of Proposition 12 with V the $\mathbf{Q}$-vector space generated by all elements of B except b. Then K_b contains $\mathbf{C}_p$, since it contains $\mathbf{Q}_p$ and is complete and algebraically closed. If $c \in B, z_c \in K_b$ iff $c \neq b$. But each K_b is algebraically closed, so no z_b can be algebraically dependent on the others over $\mathbf{C}_p$. Thus the transcendence degree of L over $\mathbf{C}_p$ exceeds the dimension of $\mathbf{R}$ over $\mathbf{Q}$ (it does not matter that we threw away one basis element), which is $2^{\aleph_0}$. On the other hand the cardinality of L is only $2^{\aleph_0}$, by Proposition 10. So the transcendence degree must equal $2^{\aleph_0}$. □

Traditionally, p-adic analysis has been done in $\mathbf{C}_p$. But every power series $F(X) = \sum_{n=0}^{\infty} a_n X^n$ with $a_n \in \mathbf{C}_p$ can be defined on L, and the radius of convergence is the same in L as in $\mathbf{C}_p$, because in either field the series converges iff the valuation of its terms approach $+\infty$. (Remember that L is complete.) As an example, we state the following proposition.

PROPOSITION 13. *There exists a unique function* $\log_p : L^* \to L$ *such that*

(1) $\log_p x = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n/n,$ *for* $v(x-1) > 0$.

(2) $\log_p xy = \log_p x + \log_p y,$ *for all* $x, y \in L^*$.

(3) $\log_p p = 0$.

Proof. The proof for L is exactly the same as the proof for $\mathbf{C}_p$. See pp. 87-88 in [7]. □

Although we can extend any power series defined on $\mathbf{C}_p$ to L, it seems that p-adic analysis rarely (if ever) would need to use properties of L not true of $\mathbf{C}_p$. All that seems important is that the field is a complete algebraically closed immediate extension of $\bar{\mathbf{Q}}_p$. It would be interesting to investigate whether anything can be gained by doing p-adic analysis in L instead of in $\mathbf{C}_p$.

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