I. Introduction

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MAXIMALLY COMPLETE FIELDS

by Bjorn POONEN

ABSTRACT. Kaplansky proved in 1942 that among all fields with a valuation having a given divisible value group G, a given algebraically closed residue field R, and a given restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$), there is one that is maximal in the strong sense that every other can be embedded in it. In this paper, we construct this field explicitly and use the explicit form to give a new proof of Kaplansky's result. The field turns out to be a Mal'cev-Neumann ring or a p-adic version of a Mal'cev-Neumann ring in which the elements are formal series of the form $\sum_{g \in S} \alpha_g p^g$ where S is a well-ordered subset of G and the α_g's are residue class representatives. We conclude with some remarks on the p-adic Mal'cev-Neumann field containing $\bar{\mathbf{Q}}_p$.

I. INTRODUCTION

It is well known that if k is an algebraically closed field of characteristic zero, then the algebraic closure of the field of Laurent series $k((t))$ is obtained by adjoining $t^{1/n}$ for each integer $n \geqslant 1$, and that the expansion of a solution to a polynomial equation over $k((t))$ can be obtained by the method of successive approximation. (For example, to find a square root of $1 + t$, one solves for the coefficients of $1, t, t^2, \ldots$ in turn.) But if k is algebraically closed of characteristic p, $\cup_{n=1}^{\infty} k((t^{1/n}))$ is no longer an algebraic closure of $k((t))$. In particular, the Artin-Schreier equation $x^p - x = t^{-1}$ has no solution in $\cup_{n=1}^{\infty} k((t^{1/n}))$. (See p. 64 of Chevalley [3].) If one attempts nevertheless to successively approximate a solution, one obtains the expansion (due to Abhyankar [1])

$$x = t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \cdots,$$

in which the exponents do not tend to ∞, as they should if the series were to converge with respect to a valuation in the usual sense. However, one checks (using the linearity of the Frobenius automorphism) that this series does formally satisfy our polynomial equation! (The other solutions are obtained by adding elements of $\mathbf{F}_p$ to this one.)

It is natural to seek a context in which series such as these make sense. If one tries to define a field containing all series $\sum_{q \in \mathbf{Q}} \alpha_q t^q$, one fails for the reason that multiplication is not well defined. But then one notices that a sequence of exponents coming from a transfinite successive approximation process must be well-ordered. If one considers only series in which the set of exponents is a well-ordered subset of $\mathbf{Q}$, one does indeed obtain a field.

Such fields are commonly known as Mal'cev-Neumann rings. (We will review their construction in Section 3.) They were introduced by Hahn in 1908, and studied in terms of valuations by Krull [8] in 1932. (Mal'cev [11] in 1948 and Neumann [12] in 1949 showed that the same construction could be performed for exponents in a non-abelian group to produce a division ring.)

If one tries to find p-adic expansions of elements algebraic over $\mathbf{Q}_p$, one encounters a similar situation. One is therefore led to construct p-adic analogues of the Mal'cev-Neumann rings. (See Section 4.) This construction is apparently new, except that Lampert [9] in 1986 described the special case of value group $\mathbf{Q}$ and residue field $\overline{\mathbf{F}}_p$ without giving details of a construction. (We will discuss this special case in detail in Section 7.)

In Section 5 we prove our main theorems. A corollary of our Theorem 2 is that a Mal'cev-Neumann ring (standard or p-adic) with divisible value group G and algebraically closed residue field R has the amazing property that every other valued field with the same value group, the same residue field, and the same restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$) can be embedded in the Mal'cev-Neumann ring. (We assume implicitly in the minimal subfield assumption that in the p-adic case the valuation of p must be the same element of G for the two fields.) Kaplansky [5] proved the existence of a field with this property using a different method. He also knew that it was a Mal'cev-Neumann ring when the restriction of the valuation to the minimal subfield is trivial, but was apparently unaware of its structure in the p-adic case.

2. PRELIMINARIES

All ordered groups G in this paper are assumed to be abelian, and we write the group law additively. We call G *divisible* if for every $g \in G$ and positive integer n, the equation $nx = g$ has a solution in G. Every ordered group can