And Lemma 1 then gives that the polynomial hull of $\bar{\mathscr{W}}$ contains $\{0\} \times \bar{V}$, with V as in the claim. The sequence of approximating polynomials converges uniformly on $\bar{V}$ to a function which provides the desired extension. The subclaim is thus proved.

Now we finish the proof of the proposition. Let φ be a function on $\mathbf{R}^+$ which is not analytic at 0, or, for the reader willing to use only the "subclaim", so that the function $(x_1, x_2) \mapsto \varphi(x_1^2 + x_2^2)$ does not have a continuous extension to $\bar{V}$, holomorphic on the interior of V, for any V intersection of a neighborhood of 0 in $\mathbf{C}^2$ with a neighborhood of $\Sigma_\varepsilon - \{0\}$. Any smooth function φ nonidentically zero but vanishing on open intervals in any neighborhood of 0 has this property. As pointed out (*), the function $(w, z_1, z_2) \mapsto \varphi(z_1^2 + z_2^2)$ in a *CR* function on $\mathscr{M}$. It follows from V or the subclaim that it is a nondecomposable one. It cannot be written as the sum of continuous boundary values of holomorphic function on wedges.

Remark. There are some few technical details (such as precising the shape of V) to be dealt with, to adapt the approach that we have just used to the case of boundary values distributions. In this setting the Baouendi Treves approximation still gives approximation by polynomials (on wedges, with locally uniform convergence, and with uniformly controlled polynomial growth when approaching the edge). Also, one can still speak about the restriction of a *CR* distribution on $\mathscr{M}$ to $\{0\} \times \mathbf{R}^2(f(0, x_1, x_2))$, (this is a basic fact used to define mini F.B.I, see [8] Corollary I.4.1.).

But it seems pointless to go into this. Indeed this kind of difficulties merely disappear when using the results explained in the next paragraph.

V. More

1) In Lemma 1, the right conclusion is in fact that 0 belongs to the interior of the polynomial hull of $\bar{W}$. Applying Lemma 1, with trivial homogeneity considerations, and replacing $\mathbf{R}_\varepsilon^2$, by $\mathbf{R}^2$, it reduces to the following proposition.

PROPOSITION 2. *Let f be a function defined on some neighborhood of 0 in $\mathbf{R}^2$. Assume that near 0, f extends holomorphically to a conic neighborhood of $\mathbf{R}^2 - \{0\}$, and also to a wedge with edge $\mathbf{R}^2$. Then f is analytic at 0.*

By conic neighborhood, we mean a cone which is a neighborhood of $\mathbf{R}^2 - \{0\}$ in $\mathbf{R}^2$. We did not make precise whether f is continuous, but we can assume it by translation in the wedge, and uniformity, (and f could as well be a distribution or a hyperfunction).

Although the result follows trivially from Trepreau's work (at least), it may have been unnoticed. Here we give a direct proof.

Proof. The analyticity of f, at 0, is related to the exponential decay of its FBI transform as $|\xi| \to +\infty$. We can make the following choice of FBI transform:

$$\mathscr{F}f(x, \xi) = \int_{|s|<R} f(s) e^{-is\cdot\xi - |\xi|(s-x)^2} ds ,$$

where $x = (x_1, x_2) \simeq 0$, $s = (s_1, s_2) \in \mathbf{R}^2$, $ds = ds_1 ds_2$, $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, $s \cdot \xi = s_1\xi_1 + s_2\xi_2$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $(s-x)^2 = (s_1 - x_1)^2 + (s_2 - x_2)^2$, and $R > 0$ is fixed (arbitrarily) (see e.g. [5] 9.6 or [8] page 416 formula (4)).

The holomorphic extendability of f, near 0, in the wedge $\mathbf{R}^2 + i\Gamma$ (Γ an open convex cone in $\mathbf{R}^2$) gives the exponential decay of $\mathscr{F}f$ as $|\xi| \to +\infty$ and $\xi \in \mathbf{R}^2 - \Gamma_0$ (Γ_0 the dual cone). We can assume $|f| \leqslant 1$, so $|\mathscr{F}f| \leqslant 1$ (taking $R < \frac{1}{2}$).

It is a trivial fact that under the hypothesis of Proposition 2, there exists Ω a conic neighborhood of $\mathbf{R}^2 - \{0\}$ so that f has near 0, a holomorphic extension to a "wedge" $\Omega + i\Gamma$. It is just the fact that in $\mathbf{R}^2$ the union of a disk centered at 0 and a cone (with vertex at 0) contains cones with vertices near 0.

Fix U an open connected neighborhood in $GL(2, \mathbf{C})$ of the real rotation matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Take U so small that $U(\mathbf{R}^2) \subset \Omega$. Consider $GL(2, \mathbf{C})$ as an open set in $\mathbf{C}^4$.

Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For every U_1 nonempty open set $U_1 \subset\subset U$, there exists $c > 0$ so that for every holomorphic function h defined on U, $|h| \leqslant 1$ one had $\log|h(e)| \leqslant c \int_{U_1} \log|h(T)|\,dT$.

On applies this estimate to

$$h(T) = \int_{|s|<R} f \circ T(s) e^{-is\cdot\xi - |\xi|(s-x)^2} ds ,$$

assuming that R is chosen small enough. It gives us that if for some nonempty open set of T's in U the FBI transform of $f \circ T$ has exponential decay in some direction, so has the FBI transform of f itself.

The key fact is now that if f extends to a wedge $\mathbf{R}^2 + i\Gamma$, $f \circ T$ extends to a totally different wedge.

For example, if T is a real rotation, $f \circ T$ extends to the wedge $\mathbf{R}^2 + iT^{-1}(\Gamma)$. Thus one gets the exponential decay of the FBI transform of f off the dual cone to the cone $T^{-1}(\Gamma)$ as well, and finally (letting T vary) the exponential decay of the FBI transform of f, as desired.

2) Now using Proposition 2 instead of Lemma 1 in IV, we are already able to prove, instead of the subclaim, that $(x_1, x_2) \mapsto f(0, x_1, x_2)$ is real analytic at $(0, 0)$.

In fact the situation is even easier, since we can now translate in the wedge we need only to know that f is defined in the wedge, without growth condition when approaching the edge.

3) To really get the real analyticity of functions which extend to a wedge in Trepreau's example we need to prove that a neighborhood of 0 in $\mathbf{C}^3$ (and not only in $\{0\} \times \mathbf{C}^2$) is included in the polynomial hull of $\bar{\mathscr{W}}$. This requires an improvement of III.

Instead of (ii) $\Rightarrow$ (i) in Lemma 3, we need to show that (ii)$'$ $\Rightarrow$ (i)$'$ where (ii)$'$ and (i)$'$ are:

$$\begin{cases} \text{(i)}' & (w_0, \zeta_1, \zeta_2) \in \hat{K} \\ \text{(ii)}' & (\zeta_1, \zeta_2) \in (\cup_{|w_0| \leqslant t} K_t)^\wedge . \end{cases}$$

Set K' be the closure of the set

$$\bigcup_{|w_0| \leqslant |w|} (\{w\} \times K_{|w|}) .$$

If (ii)$'$ holds $(0, \zeta_1, \zeta_2) \in \hat{K}'$ (by (ii) $\Rightarrow$ (i)). Using invariance under rotation in w and the local maximum principle $\hat{K}' \cap \{|w| \leqslant |w_0|\}$ is the product $\{w \in \mathbf{C}, |w| \leqslant |w_0|\} \times L$ for some compact set L in $\mathbf{C}^2$. So $(0, \zeta_1, \zeta_2) \in \hat{K}'$ implies $(w_0, \zeta_1, \zeta_2) \in \hat{K}' \subset \hat{K}$.

Remarks. It would be possible to adapt the remark at the end of the proof of Lemma 3. The change is: take $|T(\theta)| \geqslant |w_0|$ and $Q(0) = w$.

After reading this paper, J.M. Trepreau has communicated to me a proof of Proposition 2, with some common features with the proof given here, but based on a simple construction of analytic disks in $\mathbf{C} \times \mathbf{C}^2$, instead of F.B.I.