TREPREAU'S EXAMPLE, A PEDESTRIAN APPROACH

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I. Introduction

A few years ago J.M. Trepreau gave an example of a nondecomposable *CR* function, answering in the negative an outstanding question in several complex variables and microlocal analysis. The example appeared finally in print in [7], and it is magnificently explained there. Trepreau's example can also be explained by Tumanov's theory, without F.B.I., see [10].

However, I am writing this paper!

It is my goal to go through Trepreau's example with the naïvest tools (e.g. without appealing to the Hanges-Sjöstrand theorem, or to Tumanov's theory of disks). In my mind, such a basic example (in *CR* analysis and in polynomial convexity) deserves a "pedestrian" approach. Parts II-IV of the paper use only very classical tools in several complex variables and the Baouendi-Treves approximation ([3]) which is a very natural extension of Weierstrass approximation. In II, I describe what I consider to be the heart of the matter. It takes place in $\mathbf{C}^2$. In III a lifting process from $\mathbf{C}^2$ to $\mathbf{C}^3$ is discussed. It is used in IV to show, in Trepreau's example, the existence of nondecomposable *CR* functions. In V, we use F.B.I. (the simplest version suffices, and it is used in a very simple way) to complete the results given in II and III. Proposition 2 (the cone and the wedge) may be of independent interest (but cannot be claimed as being original). Finally the title of VI ("Trepreau does more") says it all!

The reader can look at [4] for the basic theory of *CR* functions, and at [1], especially §9, for positive results on wedge decomposability, background information, and further references. Notations given in II will be kept in the whole paper.

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II. The heart of the matter

II.0. Definitions, notations

Given a set $E \subset \mathbf{C}^n$, not necessarily a manifold, *a wedge* $W(= W(E, \Gamma, \rho))$ *with edge* E is defined in the following way.

For Γ a nonempty open cone in $\mathbf{C}^n$ and $\rho > 0$, one sets

$$W = \{e + \gamma \in \mathbf{C}^n, e \in E, \gamma \in \Gamma, |\gamma| < \rho\} .$$

Remark. As we will see below the words wedge and edge may be confusing. Part of the edge may very well be in the interior of the wedge, we will in fact take advantage of this situation. In case of E a germ of manifold one can instead take a cone Γ in a transverse (e.g. the normal) space, if one allows shrinking the two definitions are "equivalent". And in case E is a hypersurface, a wedge contains locally at least one of the two sides of the hypersurface.

For $\varepsilon \in [0, 1)$ set

$$\mathbf{R}^2_\varepsilon = \{(s_1 + i\varepsilon s_2, s_2 - i\varepsilon s_1) \in \mathbf{C}^2, (s_1, s_2) \in \mathbf{R}^2\} .$$

This is a tilted copy of $\mathbf{R}^2$. Set

$$\Sigma_\varepsilon = \bigcup_{0 < \varepsilon' < \varepsilon} \mathbf{R}^2_{\varepsilon'} .$$

And for $R > 0$ let

$$\Sigma^R_\varepsilon = \Sigma_\varepsilon \cap B(0, R)$$

($B(0, R)$ the open ball centered at 0, and of radius R).

II.1

The following basic (and easy) fact is at the root of Trepreau's example.

Lemma 1. *Let* W *be any wedge in* $\mathbf{C}^2$, *with edge* Σ^R_ε. *Then every point in* $\Sigma^R_\varepsilon - \{0\}$ *belongs to the interior of the polynomial hull of* $\bar{W}$.

See Proposition 2 in V for a better result. But notice that the wedge W is really needed. It is wrong that the polynomial hull of $\overline{\Sigma^R_\varepsilon}$ contains $\Sigma^R_\varepsilon - \{0\}$ in its interior. Indeed, the function $z_1^2 + z_2^2$ is real on $\Sigma_\varepsilon (z_1^2 + z_2^2 = (1 - \varepsilon^2)(s_1^2 + s_2^2))$, hence on the polynomial hull of $\overline{\Sigma^R_\varepsilon}$ (which is in fact equal to $\overline{\Sigma^R_\varepsilon}$).

Proof of Lemma 1. On Σ_ε, $z_1^2 + z_2^2 \geqslant 0$. We then foliate Σ_ε by the level sets of $z_1^2 + z_2^2$. Fix $(a, b) \in \Sigma_\varepsilon^R - \{0\}$, set $r = \sqrt{a^2 + b^2}$. Let $A = \{(z_1, z_2) \in \Sigma_\varepsilon^R, z_1^2 + z_2^2 = r^2\}$. This is an annulus in the holomorphic curve $z_1^2 + z_2^2 = r^2$, with the nonholomorphic parametrization:

$$(\varepsilon', \theta) \mapsto \frac{r}{\sqrt{1 - \varepsilon'^2}} (\cos\theta + i\varepsilon' \sin\theta, \sin\theta - i\varepsilon' \cos\theta)$$

$0 < \varepsilon' < \varepsilon_1$, $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, with $\varepsilon_1 = \min\left(\varepsilon, \sqrt{\dfrac{R^2 - r^2}{R^2 + r^2}}\right)$. (Such annuli appear in [11]). Write

$$(a, b) = (\sigma_1 + i\delta\sigma_2, \sigma_2 - i\delta\sigma_1), (\sigma_1, \sigma_2) \in \mathbf{R}^2 (0 < \delta < \varepsilon_1).$$

Let Y be the circle

$$\mathrm{Y} = \{(s_1 + i\delta s_2, s_2 - i\delta s_1) \in \mathbf{C}^2, (s_1, s_2) \in \mathbf{R}^2, s_1^2 + s_2^2 = \sigma_1^2 + \sigma_2^2\}.$$

The circle Y is entirely in the annulus A. Now, we make a trivial but crucial remark.

CLAIM. *There are points in* Y *which are in the wedge* W *(hence in the interior of the polynomial hull!).*

We now check the claim. The wedge W contains the wedge W_δ with edge $\mathbf{R}_\delta^2 \cap B(0, R)$ (given by the same cone Γ). Locally, after possible shrinking of R, there is no loss of generality in assuming that the cone Γ contains a non-zero vector $(it_1, it_2) \in i\mathbf{R}^2$. Normalize (it_1, it_2) so that $t_1^2 + t_2^2 = \sigma_1^2 + \sigma_2^2$. The point $(-t_2 + i\delta t_1, t_1 + i\delta t_2)$ is a point on the circle Y which is in the wedge W since it belongs to the ray with direction (it_1, it_2) and with origin at the point $(-t_2 + i\delta' t_1, t_1 + i\delta' t_2)$. Take $0 < \delta' < \delta$, $\delta - \delta'$ small enough.

The claim is proved, and Lemma 1 follows now by propagation along the holomorphic curve A. I wish to insist that here we are now dealing with one of the most primitive versions of propagation along holomorphic curve. It is only with the goal of having a paper elementary and as self contained as possible that I state and prove Lemma 2 to be applied to $M = \Sigma_\varepsilon^R - \{0\}$, to get Lemma 1.

LEMMA 2. *Let* M *be a (piece of)* C^1 *hypersurface in* $\mathbf{C}^2$. *Let* W *be a wedge with edge* M *(i.e. at each point of* M, W *contains at least one side of* M*). Let* C *be a holomorphic curve in* M. *If some point* *of* C *belongs to the interior of the polynomial hull of* $\bar{W}$, *then* C *is entirely included in the interior of the polynomial hull of* $\bar{W}$.

By holomorphic curve we will mean a connected 1-dimensional holomorphic manifold.

Proof. Let O be the interior of the polynomial hull of $\bar{W}$. It has to be shown that the set of points $p \in C$ which belong to O is closed in C. It is obviously open. Things being so localized one has to face the following situation: a "small" analytic disk given by a holomorphic parametrization $\varphi : \bar{\Delta} \to C$ (Δ the unit disk in $\mathbf{C}$) so that $\varphi(1) \in O$, U^+ a side of M included in W (at least one of the two sides is such) hence in O, in some neighborhood of $\varphi(\bar{\Delta})$; and one has to show that $\varphi(0) \in O$. Fix ψ a holomorphic map from $\mathbf{C}$ into $\mathbf{C}^n$ so that: $\psi(e^{i\theta}) \simeq -\vec{N}$ for θ outside some small neighborhood of 0 (mod 2π), where $\vec{N}$ is the unit outer normal to M (with respect to U^+), at say the point $\varphi(0)$, and $\psi(0)$ is arbitrarily chosen.

For $\eta > 0$, η small enough $\varphi(e^{i\theta}) + \eta\psi(e^{i\theta}) \in O$ for all θ, hence $\varphi(0) + \eta\psi(0) \in O$. Taking into account some uniformity with respect to $\psi(0)$, this gives Lemma 2.

III. Lifting to $\mathbf{C}^3$

We are simply going to consider sets K in $\mathbf{C}^3$ rotationally invariant in the first variable, that we describe as follows. For each $t \in [0, t_0]$ we are given a compact set $K_t \subset \mathbf{C}^2$. We consider the set $K \subset \mathbf{C}^3$ which is the closure of the set $\{(w, z_1, z_2) \in \mathbf{C}^3 ; (z_1, z_2) \in K_{|w|}, |w| \leqslant t_0\}$. i.e.

$$K = \overline{\bigcup_{|w| \leqslant t_0} \{w\} \times K_{|w|}} .$$

$\hat{K}$ denotes the polynomial hull of K in $\mathbf{C}^3$, while $\widehat{\cup K_t}$ denotes the polynomial hull in $\mathbf{C}^2$ of the closure of the set $\bigcup_{t \leqslant t_0} K_t$.

LEMMA 3. *Let* $(0, \zeta_1, \zeta_2) \in \mathbf{C}^3$, *the following are equivalent:*

$$\begin{cases} (i) & (0, \zeta_1, \zeta_2) \in \hat{K} \\ (ii) & (\zeta_1, \zeta_2) \in \widehat{\cup K_t} . \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) is trivial. We are interested in (ii) $\Rightarrow$ (i). Let $P(w, z_1, z_2)$ be a polynomial in 3 variables. To P we associate the polynomial $\tilde{P}$ defined by

$$\tilde{P}(w, z_1, z_2) = P(0, z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta} w, z_1, z_2) d\theta .$$

Since K is invariant under rotation in the w variable:

$$\sup_K |\tilde{P}| \leqslant \sup_K |P|.$$

Set $Q(z_1, z_2) = P(0, z_1, z_2)$. Using (ii) one gets

$$|P(0, \zeta_1, \zeta_2)| = |Q(\zeta_1, \zeta_2)| \leqslant \sup_{\cup K_t} |Q| = \sup_K |\tilde{P}| \leqslant \sup_K |P|.$$

So (i) is established.

Remark. There is another approach to Lemma 3, which may better "explain" the situation, and that we just sketch. If $\varphi : \Delta \to \mathbf{C}^2$ is a holomorphic disk (φ continuous on $\bar{\Delta}$, holomorphic on Δ) and T is a continuous map from $\mathbf{R}/2\pi\mathbf{Z}$ into $[0, t_0]$ so that $\varphi(e^{i\theta}) \in K_{T(\theta)}$ ($\theta \in [0, 2\pi)$), then $\varphi(0) \in \widehat{\cup K_t}$. One sees that $(0, \varphi(0)) \in \hat{K}$ by considering holomorphic disks $(Q, \varphi) : \Delta \to \mathbf{C} \times \mathbf{C}^2$, with $Q(0) = 0$ and $|Q(e^{i\theta})| \simeq T(\theta)$. Carrying this out in general may require the use of the fundamental theorem by Poletsky [6], which says that, in an appropriate sense, polynomial hulls are always explained by holomorphic disks.

IV. Trepeau's example

Here we describe a class of examples. Let χ be a smooth real valued function defined on $[0, 1]$, constant in no neighborhood of 0, and so that $\chi(0) = 0$, $|\chi| < 1$. In one of the versions of Trepreau's original example $\chi(t) = t$. Let $\mathscr{M}$ be the generic 4-dimensional manifold in $\mathbf{C}^3$, given by:

$$\mathscr{M} = \{(w, z_1, z_2) \in \mathbf{C}^3,\ |w| < 1, z_1 = s_1 + i\chi(|w|^2)s_2,$$
$$z_2 = s_2 - i\chi(|w|^2)s_1;\ (s_1, s_2) \in \mathbf{R}^2\}.$$

Notice that on $\mathscr{M}$, $z_1^2 + z_2^2$ is a real valued function, (on $\mathscr{M}$, $z_1^2 + z_2^2 \geqslant 0$), hence:

(*) *Any function which depends only on* $(z_1^2 + z_2^2)$ *is a CR function on* $\mathscr{M}$.

This already gives example of *CR* functions which cannot be holomorphically extended to any wedge. The existence of such functions is related to the fact that $\mathscr{M}$ is not "minimal" (in the sense of Tumanov), it contains $\mathbf{C} \times \{0\} \times \{0\}$ as a (nongeneric) *CR* manifold of same *CR* dimension (see [9], [2]).

Before going any further, we wish to rewrite the definition of $\mathscr{M}$ in the spirit of II and III. With the notations used in II:

$$\mathscr{M} = \{(w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, (z_1, z_2) \in \mathbf{R}^2_{\chi(|w|^2)}\} .$$

PROPOSITION 1. *There are smooth CR functions on* $\mathscr{M}$ *which in no neighborhood of* 0 *can be decomposed into the sum of boundary values of functions holomorphic in wedges (with edge* $\mathscr{M}$*).*

There is some ambiguity in the statement since it is not made precise in which sense boundary values are taken. To keep things at the most elementary level we will treat in detail the case of *continuous boundary values.* See the remark below, and V, for the case of more general boundary values. (Although I suspect that one can prove, as a general fact, that if every *smooth CR* function is decomposable, then the decomposition can be done with functions continuous (and even smooth) up to the edge).

Proof. We can assume that in any neighborhood of 0, χ takes some strictly positive values (permuting the variables z_1 and z_2, if needed). Let $\mathscr{W}$ be an arbitrary wedge, with edge $\mathscr{M}_0$ the intersection of $\mathscr{M}$ with some neighborhood of 0.

The reader willing to read V will see *that every CR function on* $\mathscr{M}_0$ *which has a holomorphic extension to some wedge with edge* $\mathscr{M}_0$ *is analytic in some neighborhood of* 0.

The reader unwilling to read V, and willing to use only the simple techniques used in II and III will have to use the "subclaim".

"SUBCLAIM". *Let* f *be a continuous CR function on* $\mathscr{M}_0$, *which has a holomorphic extension to* $\mathscr{W}$. *Then there exists* $\varepsilon > 0$, *and* V *the intersection of a neighborhood of* 0 *in* $\mathbf{C}^2$ *with a neighborhood of* $\Sigma_\varepsilon - \{0\}$ *so that the function* $(x_1, x_2) \mapsto f(0, x_1, x_2)$ *has a continuous extension to* $\bar{V}$, *holomorphic on the interior of* V.

Proof of the subclaim. After shrinking of $\mathscr{W}$ and $\mathscr{M}_0$, the Baouendi-Treves approximation formula ([3], [8] II.2) shows that f is the uniform limit on $\bar{\mathscr{W}}$ of a sequence of polynomials (P_j).

For Γ an open cone in $\mathbf{C}^2$ and $\rho > 0$, and $w \in \mathbf{C} \mid |w| < \rho$, we consider $K_{|w|}$ the closure in $\mathbf{C}^2$ of the wedge $W(\mathbf{R}^2_{\chi(|w|^2)} \cap B(0, \rho), \Gamma, e)$, (with edge in $\mathbf{R}^2_{\chi(|w|^2)}$). One can choose Γ and ρ so that for every $w \in \mathbf{C}, |w| < \rho$, we have $\{w\} \times K_{|w|} \subset \mathscr{W}$. We apply Lemma 3 to these sets $K_{|w|}$ and to the set $K = \cup(\{w\} \times K_{|w|})$.

And Lemma 1 then gives that the polynomial hull of $\bar{\mathscr{W}}$ contains $\{0\} \times \bar{V}$, with V as in the claim. The sequence of approximating polynomials converges uniformly on $\bar{V}$ to a function which provides the desired extension. The subclaim is thus proved.

Now we finish the proof of the proposition. Let φ be a function on $\mathbf{R}^+$ which is not analytic at 0, or, for the reader willing to use only the "subclaim", so that the function $(x_1, x_2) \mapsto \varphi(x_1^2 + x_2^2)$ does not have a continuous extension to $\bar{V}$, holomorphic on the interior of V, for any V intersection of a neighborhood of 0 in $\mathbf{C}^2$ with a neighborhood of $\Sigma_\varepsilon - \{0\}$. Any smooth function φ nonidentically zero but vanishing on open intervals in any neighborhood of 0 has this property. As pointed out (*), the function $(w, z_1, z_2) \mapsto \varphi(z_1^2 + z_2^2)$ in a *CR* function on $\mathscr{M}$. It follows from V or the subclaim that it is a nondecomposable one. It cannot be written as the sum of continuous boundary values of holomorphic function on wedges.

Remark. There are some few technical details (such as precising the shape of V) to be dealt with, to adapt the approach that we have just used to the case of boundary values distributions. In this setting the Baouendi Treves approximation still gives approximation by polynomials (on wedges, with locally uniform convergence, and with uniformly controlled polynomial growth when approaching the edge). Also, one can still speak about the restriction of a *CR* distribution on $\mathscr{M}$ to $\{0\} \times \mathbf{R}^2(f(0, x_1, x_2))$, (this is a basic fact used to define mini F.B.I, see [8] Corollary I.4.1.).

But it seems pointless to go into this. Indeed this kind of difficulties merely disappear when using the results explained in the next paragraph.

V. More

1) In Lemma 1, the right conclusion is in fact that 0 belongs to the interior of the polynomial hull of $\bar{W}$. Applying Lemma 1, with trivial homogeneity considerations, and replacing $\mathbf{R}^2_\varepsilon$, by $\mathbf{R}^2$, it reduces to the following proposition.

Proposition 2. *Let* f *be a function defined on some neighborhood of* 0 *in* $\mathbf{R}^2$. *Assume that near* 0, f *extends holomorphically to a conic neighborhood of* $\mathbf{R}^2 - \{0\}$, *and also to a wedge with edge* $\mathbf{R}^2$. *Then* f *is analytic at* 0.

By conic neighborhood, we mean a cone which is a neighborhood of $\mathbf{R}^2 - \{0\}$ in $\mathbf{R}^2$. We did not make precise whether f is continuous, but we can assume it by translation in the wedge, and uniformity, (and f could as well be a distribution or a hyperfunction).

Although the result follows trivially from Trepreau's work (at least), it may have been unnoticed. Here we give a direct proof.

Proof. The analyticity of f, at 0, is related to the exponential decay of its FBI transform as $|\xi| \to +\infty$. We can make the following choice of FBI transform:

$$\mathscr{F}f(x, \xi) = \int_{|s|<R} f(s) e^{-is\cdot\xi - |\xi|(s-x)^2} ds ,$$

where $x = (x_1, x_2) \simeq 0$, $s = (s_1, s_2) \in \mathbf{R}^2$, $ds = ds_1 ds_2$, $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, $s \cdot \xi = s_1\xi_1 + s_2\xi_2$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $(s-x)^2 = (s_1 - x_1)^2 + (s_2 - x_2)^2$, and $R > 0$ is fixed (arbitrarily) (see e.g. [5] 9.6 or [8] page 416 formula (4)).

The holomorphic extendability of f, near 0, in the wedge $\mathbf{R}^2 + i\Gamma$ (Γ an open convex cone in $\mathbf{R}^2$) gives the exponential decay of $\mathscr{F}f$ as $|\xi| \to +\infty$ and $\xi \in \mathbf{R}^2 - \Gamma_0$ (Γ_0 the dual cone). We can assume $|f| \leqslant 1$, so $|\mathscr{F}f| \leqslant 1$ (taking $R < \frac{1}{2}$).

It is a trivial fact that under the hypothesis of Proposition 2, there exists Ω a conic neighborhood of $\mathbf{R}^2 - \{0\}$ so that f has near 0, a holomorphic extension to a "wedge" $\Omega + i\Gamma$. It is just the fact that in $\mathbf{R}^2$ the union of a disk centered at 0 and a cone (with vertex at 0) contains cones with vertices near 0.

Fix U an open connected neighborhood in $GL(2, \mathbf{C})$ of the real rotation matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Take U so small that $U(\mathbf{R}^2) \subset \Omega$. Consider $GL(2, \mathbf{C})$ as an open set in $\mathbf{C}^4$.

Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For every U_1 nonempty open set $U_1 \subset\subset U$, there exists $c > 0$ so that for every holomorphic function h defined on U, $|h| \leqslant 1$ one had $\log|h(e)| \leqslant c \int_{U_1} \log|h(T)|dT$.

On applies this estimate to

$$h(T) = \int_{|s|<R} f \circ T(s) e^{-is\cdot\xi - |\xi|(s-x)^2} ds ,$$

assuming that R is chosen small enough. It gives us that if for some nonempty open set of T's in U the FBI transform of $f \circ T$ has exponential decay in some direction, so has the FBI transform of f itself.

The key fact is now that if f extends to a wedge $\mathbf{R}^2 + i\Gamma$, $f \circ T$ extends to a totally different wedge.

For example, if T is a real rotation, $f \circ T$ extends to the wedge $\mathbf{R}^2 + iT^{-1}(\Gamma)$. Thus one gets the exponential decay of the FBI transform of f off the dual cone to the cone $T^{-1}(\Gamma)$ as well, and finally (letting T vary) the exponential decay of the FBI transform of f, as desired.

2) Now using Proposition 2 instead of Lemma 1 in IV, we are already able to prove, instead of the subclaim, that $(x_1, x_2) \mapsto f(0, x_1, x_2)$ is real analytic at $(0, 0)$.

In fact the situation is even easier, since we can now translate in the wedge we need only to know that f is defined in the wedge, without growth condition when approaching the edge.

3) To really get the real analyticity of functions which extend to a wedge in Trepreau's example we need to prove that a neighborhood of 0 in $\mathbf{C}^3$ (and not only in $\{0\} \times \mathbf{C}^2$) is included in the polynomial hull of $\bar{\mathscr{W}}$. This requires an improvement of III.

Instead of (ii) $\Rightarrow$ (i) in Lemma 3, we need to show that (ii)$'$ $\Rightarrow$ (i)$'$ where (ii)$'$ and (i)$'$ are:

$$\begin{cases} \text{(i)}' & (w_0, \zeta_1, \zeta_2) \in \hat{K} \\ \text{(ii)}' & (\zeta_1, \zeta_2) \in (\cup_{|w_0| \leqslant t} K_t)^\wedge \,. \end{cases}$$

Set K' be the closure of the set

$$\bigcup_{|w_0| \leqslant |w|} (\{w\} \times K_{|w|}) \,.$$

If (ii)$'$ holds $(0, \zeta_1, \zeta_2) \in \hat{K}'$ (by (ii) $\Rightarrow$ (i)). Using invariance under rotation in w and the local maximum principle $\hat{K}' \cap \{|w| \leqslant |w_0|\}$ is the product $\{w \in \mathbf{C}, |w| \leqslant |w_0|\} \times L$ for some compact set L in $\mathbf{C}^2$. So $(0, \zeta_1, \zeta_2) \in \hat{K}'$ implies $(w_0, \zeta_1, \zeta_2) \in \hat{K}' \subset \hat{K}$.

Remarks. It would be possible to adapt the remark at the end of the proof of Lemma 3. The change is: take $|T(\theta)| \geqslant |w_0|$ and $Q(0) = w$.

After reading this paper, J.M. Trepreau has communicated to me a proof of Proposition 2, with some common features with the proof given here, but based on a simple construction of analytic disks in $\mathbf{C} \times \mathbf{C}^2$, instead of F.B.I.

VI. TREPREAU DOES MORE

1) Trepreau not only deals with wedge decomposability (which is related to wave front sets in acute cones), he shows in his example that the wave front set of a *CR* function at 0 is either empty or the whole conormal.

2) Our proof may not adapt to a slightly perturbed situation. But this is precisely the point! Let us compare with the theory of elliptic points for surfaces with isolated complex tangencies. The model case ($z_2 = |z_1|^2 + \alpha \operatorname{Re} z_1^2, 0 \leqslant \alpha < 1$) is totally trivial to explore. Only the perturbed case needs Bishop's disks. We hope that the reader is convinced that the same is true here.

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