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# PONCELET'S THEOREM AND DUAL BILLIARDS

by Serge TABACHNIKOV

While in captivity as a prisoner of war in the Russian city of Saratov, from winter 1812-13 till June 1814, Jean-Victor Poncelet discovered his celebrated closure theorem. Its statement is as follows. Given two nested ellipses $\Gamma_0$, $\Gamma_1$ in the plane, one plays the game illustrated in figure 1: choose a point $x$ on $\Gamma_1$, draw a tangent line to $\Gamma_0$ from $x$, find its intersection $y$ with $\Gamma_1$ and iterate, taking $y$ as a new starting point. The claim is that, if $x$ returns back to the initial position after a finite number of iterations, then every point of $\Gamma_1$ will return back after the same number of iterations.

The prehistory of Poncelet's theorem is related to works of many mathematicians (Euler and Steiner among them). The original proof, given by Poncelet, was synthetic and rather complicated. Jacobi soon recognized a relation between Poncelet's theorem and the theory of elliptic functions, and published his analytic proof in 1828. In 1853 Cayley gave explicit conditions for Poncelet polygons to be closed. A modern conceptual proof of the theorem and its generalization to 3-space were found by Griffith and Harris some 15 years ago. We refer to [B.-K.-O.-R.] and [B], chap. 16.6 for the history of Poncelet's theorem and its classical proofs; and to [G.-H. 1, 2] for the modern one.

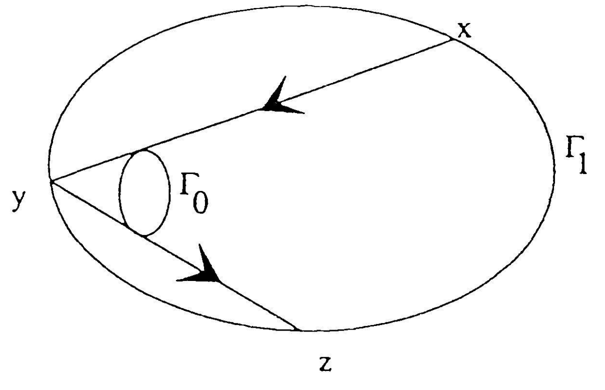


Fig.1

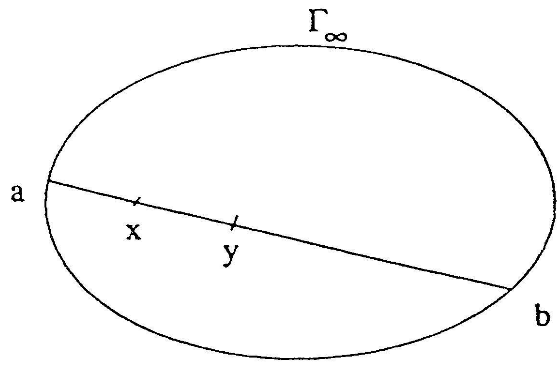


Fig.2

Here we give still another proof of Poncelet's theorem. Given two nested ellipses $\Gamma_0$, $\Gamma_\infty$, we define an area form in the annulus $A$ between them (so that the total area of $A$ is infinite), and an area-preserving map $T$ of $A$ into itself. This map is integrable in the sense that $A$ is foliated by $T$-invariant curves $\Gamma_\lambda$. A standard argument shows then that each $\Gamma_\lambda$ carries an affine structure and $T$, restricted to $\Gamma_\lambda$, is a shift $t \to t + c$. It follows that, if $T$ has an $N$-periodic orbit on $\Gamma_\lambda$, then all its orbits are $N$-periodic.

To define the area form and the map $T$ identify the interior of $\Gamma_\infty$ with the hyperbolic plane (the Klein-Beltrami model). The distance between points is given by $\operatorname{dist}(x, y) = |\log[x, y, b, a]|$, where [ ] denotes the cross-ratio — see figure 2. The area form is the hyperbolic area, and the map $T$ is the dual billiard transformation with respect to $\Gamma_0$, defined as follows (see [M], [T1, 2] for more information on this map). Given a point $x$ outside of $\Gamma_0$, draw a tangent line to $\Gamma_0$ through $x$ (say, the right one from the view-point of $x$) and reflect $x$ in the point of tangency $W$ — see figure 3. Thus, $\operatorname{dist}(x, W) = \operatorname{dist}(W, Tx)$, where the distances are those in the hyperbolic metric.

We need two properties of the dual billiard map.

PROPOSITION 1. *For any convex smooth curve* $\Gamma_0$ *(not necessarily an ellipse) the map* $T$ *preserves hyperbolic area.*

Two conics $\Gamma_0$ and $\Gamma_\infty$ determine a pencil of conics $\Gamma_\lambda$, passing through the four intersection points $\Gamma_0 \cap \Gamma_\infty$ (imaginary in our case). The curves $\Gamma_\lambda$ foliate the annulus $A$.

PROPOSITION 2. *The map* $T$ *preserves each ellipse of the pencil in* $A$.

We will prove the propositions a little later; now we deduce Poncelet's theorem from them.

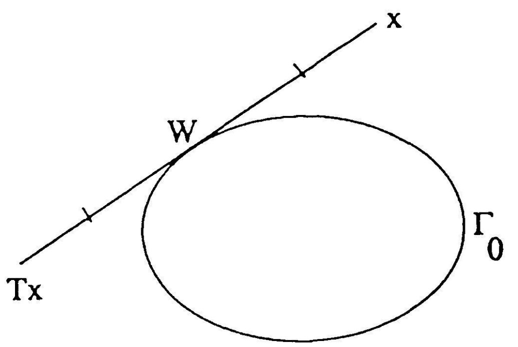


Fig.3

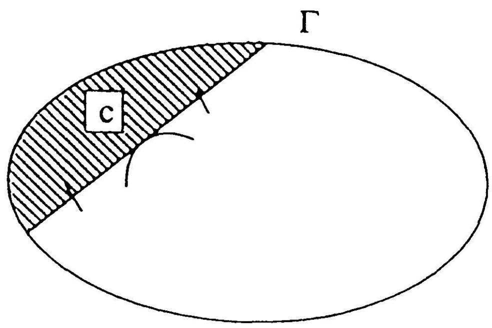


Fig.4

Recall that an affine structure on a curve is an atlas on it such that changes of charts are local affine transformations of the real line. Recall also that given a plane domain with an area form $\omega$, a smooth function $f$ in the domain has the symplectic gradient $v$ associated to it (it is also called the Hamiltonian vector field with the Hamiltonian function $f$). In local coordinates $(p, q)$ such that $\omega = dp \wedge dq$, one has $v = -\dfrac{\partial f}{\partial q}\dfrac{\partial}{\partial p} + \dfrac{\partial f}{\partial p}\dfrac{\partial}{\partial q}$.

Given two initial ellipses $\Gamma_0$, $\Gamma_1$, include them into a pencil and take $\Gamma_\infty$ to be an ellipse from this pencil outside of $\Gamma_1$. Consider the dual billiard map in the annulus $A$ between $\Gamma_0$ and $\Gamma_\infty$. Then $A$ is foliated by invariant curves, and $\Gamma_1$ is one of them. Define an affine structure on $\Gamma_1$. Let $f$ be a smooth function in a neighbourhood of $\Gamma_1$, which is constant on each curve $\Gamma_\lambda$. Its symplectic gradient $v$ is tangent to $\Gamma_1$. If one replaces $f$ by another function $g = \phi(f)$, then the corresponding vector field on $\Gamma_1$ becomes $u = \phi'(f(\Gamma_1))v$. Hence a vector field is defined on $\Gamma_1$ up to a multiplication by a constant. Fix $v$ requiring that its time-one map is the identity. Let $t$ mod 1 be the coordinate on $\Gamma_1$, such that $v = \partial/\partial t$. The parameter $t$ is defined up to a shift $t \to t + a$ and it determines an affine structure on $\Gamma_1$. Since $T$ preserves the area form and the foliation $\Gamma_\lambda$ leaf-wise, it is an affine transformation on $\Gamma_1$. Its degree is one, hence it is a shift $t \to t + c$. Poncelet's theorem follows.

The above argument is a particular case of a more general well-known consideration. Given a Lagrangian foliation $\mathscr{F}$ on a symplectic manifold $(M, \omega)$, its leaves carry a canonical affine structure. This structure is defined on a leaf $F$ by a locally free action of the additive group of the cotangent space $T_F^*(M/\mathscr{F})$, where $M/\mathscr{F}$ is the (locally defined) space of leaves of the foliation. Namely, functions on $M/\mathscr{F}$, considered as Hamiltonians on $M$, define there commuting vector fields which are tangent to the leaves of $\mathscr{F}$. A function with zero differential at $F \in M/\mathscr{F}$ defines the zero field on $F$. If in addition a symplectic map is given, preserving $\mathscr{F}$ leaf-wise, then its restriction to each leaf is an affine transformation therein. If, moreover, the leaves are tori, then the restriction of the map to each leaf is a shift (Arnold-Liouville theorem; see, e.g. [A.-G.]).

Now we prove the two propositions.

LEMMA 3. *Given a smooth convex curve* $\Gamma$ *in the hyperbolic plane, fix a number* $c > 0$ *and consider the family of straight lines which cut off segments of areas* $c$ *from* $\Gamma$. *Then the envelope of this family bisects each segment of the family at the point of tangency (see figure 4).*

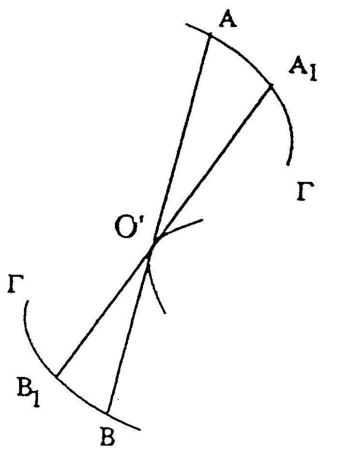


Fig.5

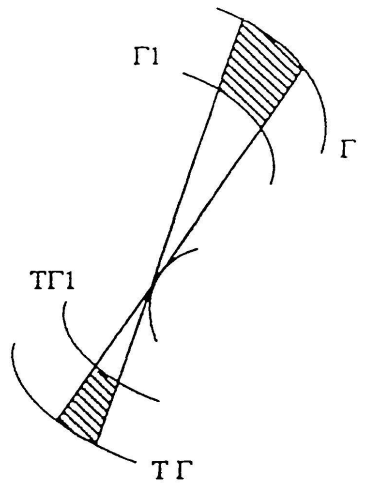


Fig.6

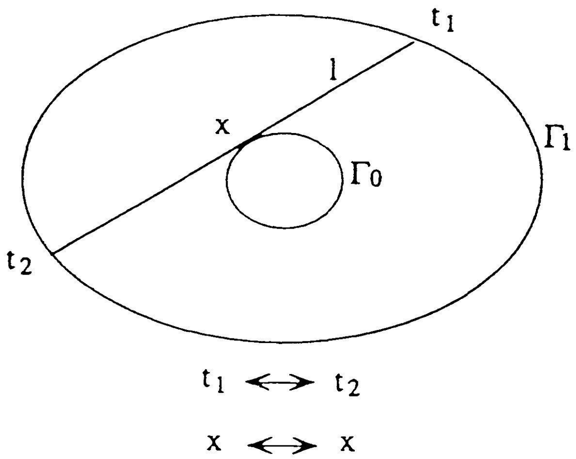


Fig.7

*Proof* (see [F.-T.] for the Euclidean case). Let $AB$ be a line from the family and $O$ its tangency point to the envelope. Assume that $AO > BO$ — see figure 5. Let $A_1B_1$ be a sufficiently close line from the family and $O' = AB \cap A_1B_1$. We have: $\text{Area}(AO'A_1) = \text{Area}(BO'B_1)$. Also $AO' > BO'$ and $A_1O' > B_1O'$. Then the central symmetry in $O'$ sends the "triangle" $BO'B_1$ inside $AO'A_1$; a contradiction.

Proposition 1 follows: the shaded areas in figure 6 are equal.

Consider now a pencil of conics $\Gamma_\lambda$ and let $l$ be a line in the plane. Intersections of $l$ with the conics $\Gamma_\lambda$ define an involution on $l$ — see figure 7.

LEMMA 4 (Desargues' Theorem. [B], 14.2.8.3). *This involution is a projective transformation of the line $l$.*

*Proof.* Applying a projective transformation of the plane, we make the conics $\Gamma_\lambda$ concentric. Take the center as the origin of the plane $V$. Then the pencil consists of conics.

$$\Gamma_\lambda = \{ < A_\lambda(x),\ x > \ = 1 \}$$

where $A_\lambda = A + \lambda E$ and $A, E\colon V \to V^*$ are selfadjoint operators. Let $l$ be tangent to $\Gamma_0$ at $x$ and $u$ be the tangent vector there. Parametrize $l$ by a real $t$ such that points of $l$ are $x + tu$.

The intersection $l \cap \Gamma_\lambda$ is given by

$$< (A + \lambda E)\ (x + tu),\ x + tu > \ = 1\ .$$

Since $< Ax, x > \ = 1$ and $< Ax, u > \ = 0$, we simplify to

$$< (A + \lambda E) u, u > t^2 + 2\lambda t < Ex, u > + \lambda < Ex, x > \ = 0\ .$$

It follows that

$$\frac{1}{t_1} + \frac{1}{t_2} = -2\, \frac{< Ex, u >}{< Ex, x >}$$

independently of $\lambda$; here $t_1$ and $t_2$ are the roots. Hence the correspondence $t_1 \leftrightarrow t_2$ is fractional-linear, that is, projective.

Note, that the two fixed points of the involution are the tangency points of $l$ to conics of the pencil (one of them is an ellipse, another- a hyperbola). There are exactly two such tangency points by the Jacobi theorem.

To deduce proposition 2, consider figure 8. The involution sends $x$ to $y$ and $U$ to $V$, preserving $W$. Since it preserves the cross-ratio, we have: $\mathrm{dist}(x, W) = \mathrm{dist}(W, y)$.

One can deduce a little more from these proofs. Consider three nested ellipses from a pencil: $\Gamma''$, $\Gamma'$ and $\Gamma$, and identify the interior of the outer one with the hyperbolic plane. Let $T'$ and $T''$ be the dual billiard transformations in the hyperbolic plane, associated to $\Gamma'$ and $\Gamma''$.

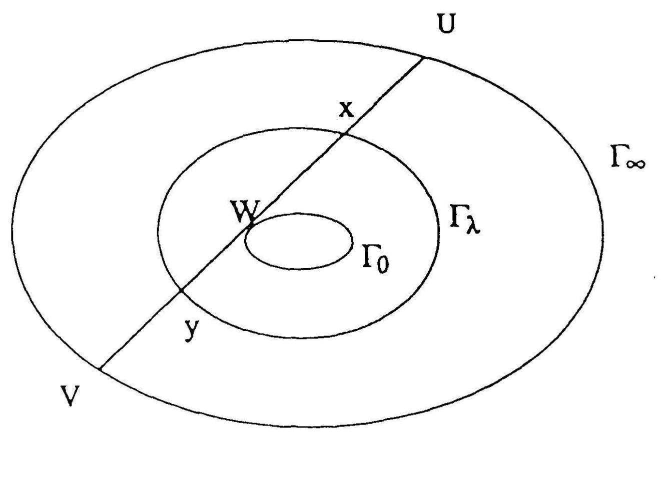


Fig.8

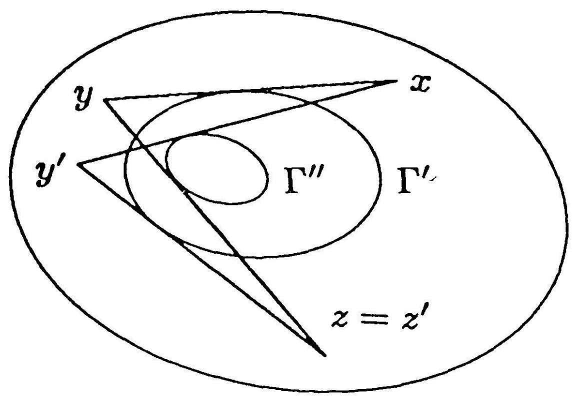


Fig.9

PROPOSITION 5. $T'T'' = T''T'$ (see figure 9).

*Proof.* For any point of the annulus between $\Gamma'$ and $\Gamma$ there is an ellipse from the pencil through it. Both maps preserve this ellipse and both are shifts in the affine parameter on it. Since shifts commute, the proposition follows.

We refer to [T3] for a partial converse statement to the proposition.

One can also slightly generalize Poncelet's theorem (this generalization was known to Poncelet too). Consider a number of ellipses from a pencil: $\Gamma, \Gamma', \Gamma''$ etc, and let $\Gamma$ contain all the others. Choose a point $x \in \Gamma$, draw a tangent line to $\Gamma'$, find its intersection with $\Gamma$, draw a tangent line to $\Gamma''$ etc. Then, if $x$ returns back after a number of iterations, any initial point on $\Gamma$ does. It follows again from the fact that $T', T''$ etc. are shifts in the affine parameter on $\Gamma$.

We conclude with a conjecture. Let a smooth strictly convex curve $\Gamma_0$ be given in the plane. Assume that its outer neighbourhood is foliated by convex curves $\Gamma_\lambda, \lambda \in [0, \varepsilon[$. Assume also that, for any line tangent to $\Gamma_0$, the (local) involution, defined on it by its intersections with the curves $\Gamma_\lambda$, is a projective transformation. Then the curves $\Gamma_\lambda$ belong to a pencil of ellipses.

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