§10. The complex of geometric forms on a curve in R^2

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and are linear fractional images of the equiangular spiral with angle $\pm\pi/4$ given by

$$\sigma(v) = e^{(1\pm i)v} = e^{v}e^{\pm iv}\,.$$

We note in particular that the inverse length of "one loop" of the inversive geodesic is 2π. Figure 9.1 is a picture of one such loop.

Equiangular spirals have two accumulation points, the poles, one at the origin and the other at infinity. These poles determine the family of circles through them (straight lines in this case) as well as a second family of circles orthogonal to the first. The equiangular spiral meets each family in fixed angles. The same is true for linear fractional images of this configuration, and with the same angles.

The connection between Q and the angle $\varphi \in (0, \pi/2)$ between the loxodrome and its first family of circles is given by

$$\tan\varphi = Q + \sqrt{Q^2+1}\,.$$

In figure 9.2, we show the inversive geodesic with poles at ± 1 together with its first family of circles.

In figure 9.3 we see the loxodrome again, in a perspective view this time, thrown up onto the two-sphere by the inverse of stereographic projection, along with its second family of circles.

We remark that it seems to be impossible to show the inverse geodesic in such a way as to allow more than one or two loops to appear to the eye, while at the same time allowing no distortion of the figure. This may account for a number of distorted diagrams of this loxodrome which have appeared in the literature. Of course one can picture many loops of some equiangular spirals, say with $Q \gg 0$. At the other extreme with $Q \ll 0$ we have a circumstance for which, in any scale, the corresponding equiangular spiral appears to the eye to be a straight line issuing from the origin. However as one "zooms" in or out this "straight line" appears to rotate about the origin.

§10. THE COMPLEX OF GEOMETRIC FORMS ON A CURVE IN $\mathbf{R}^2$

Among the various forms on a curve in $\mathbf{R}^2$, some, such as ω and Q, can be thought of as arising from the local way in which the curve is embedded in $\mathbf{R}^2$; that is they arise from the local geometric nature of the embedding and are invariant under Möbius transformations. These are the "smooth local geometric forms" of inversive geometry, or "geometric forms" for short. To be more precise, suppose that $p_j \in \gamma_j (j=1,2)$, where at p_j the germ of γ_j has a local description of the form $\gamma_j = \{z \in R^2 | F_j(z) = 0\}$. We say (γ_1, p_1) and (γ_2, p_2) have contact of order at least r if for some choice of the F_j there is a Möbius transformation $g \in G$ moving p_1 to p_2 in such a way that the Taylor series in x and y for $F_2(x,y)$ and $F_1 \circ g(x,y)$ are equivalent in total degrees $\leqslant r$. We define a geometric form η of dimension d and order r to be an assignment $\gamma \mapsto \eta_\gamma$ which attaches to each vertex free curve γ a d-form η_γ on it such that the assignment satisfies:

i) *Invariance.* If $p_j \in \gamma_j (j=1,2)$ have contact of order at least r via the element $g \in G$, then

$$\eta_{\gamma_1}(p_1) = g^*(\eta_{\gamma_2}(p_2))$$

ii) *Smoothness.* If $\gamma(t_1, t_2, \ldots t_k)$ is a smooth k-parameter family of curves with parametrization $\sigma(t, t_1, t_2, \ldots t_k)$, then the function

$$\eta_{\gamma(t_1,t_2,\ldots t_k)} \quad \text{if} \quad d = 0, \quad \text{or} \quad \eta_{\gamma(t_1,t_2,\ldots t_k)}\left(\frac{\partial \sigma}{\partial t}\right) \quad \text{if} \quad d = 1$$

depends smoothly on $t, t_1, t_2, \ldots t_k$.

The following lemma relates inversive curvature to contact.

LEMMA 10.1. *There are universal polynomials* $P_r(v_0, v_1, v_2, \ldots v_{r-5}) \in \mathbf{Q}[v_0, v_1, v_2, \ldots v_{r-5}]$ *for every* $r \geqslant 5$ *with the property that at any non-vertex point* p *on a curve* γ *with inversive curvature function* Q, *the pair* (γ, p) *has contact of order* $\geqslant r$ *with the curve*

$$y = \frac{x^3}{6} + t_5 x^5 + t_6 x^6 + \ldots + t_r x^r \tag{10.2}$$

at the origin, where $t_k = P_k(Q, Q^{(1)}, Q^{(2)}, \ldots Q^{(k-5)})(p)$ *for* $k = 5, 6, \ldots, r$.

Proof. We regard the coefficients t_k as functions on the curve γ; that is, for each non-vertex point p on the curve, there is a unique inversive transformation sending p to the origin and throwing the curve into the form 10.2 (cf. §6) and so the coefficients $t_k(p)$ are uniquely determined by the curve γ and the point p. Thus the t_k's are functions on the curve; for example $t_5 = Q/60$ by the results of §6. By the formula 6.1 we see that $\kappa' = 1$ at the origin for the curve 10.2 which implies that

$$\frac{dt_k}{dv} = \frac{dt_k}{dx}$$

so that x can be used as inversive arc-length parameter to first order for the curve at the origin. Given the t_k's at the origin we can try to calculate them at a nearby point on the curve $(h, 0) + O(h^2)$. Displacing this point to the origin yields the following expression for the translated curve

$$y = \frac{(x+h)^3}{6} + \sum_{k=5}^{r} t_k(x+h)^k + O((x+h)^{r+1}) .$$

Let I_k be the ideal generated by $t_5, t_6, \dots t_{k-1}, x^k, h^2$, so that this equation implies

$$y = \frac{h}{2} x^2 + \frac{x^3}{6} + hkt_k x^{k-1} \mod I_k .$$

Then the substitution

$$z \mapsto \frac{z}{1 - i\frac{h}{2} z} = z + i\frac{h}{2} z^2 + O(h^2)$$

throws the equation into the form

$$y = \frac{x^3}{6} - \frac{5}{72} hx^6 + hkt_k x^{k-1} \mod I_k .$$

Since there is no quartic term mod I_k, this is already the normal form we seek, and we have shown that

$$t_{k-1}(h) = hkt_k(0) + A + Bh + O(h^2), \quad \text{where} \quad A, B \in \mathbf{Q}[t_5, t_6, \dots, t_{k-1}] .$$

Thus

$$\frac{dt_{k-1}}{dv} = kt_k + B$$

and hence

$$t_k \in \mathbf{Q}\left[t_5, t_6, \dots, t_{k-1}, \frac{dt_{k-1}}{dv}\right] .$$

Since $t_5 = Q/60$ it follows inductively that

$$t_k \in \mathbf{Q}[Q, Q^{(1)}, \dots, Q^{(k-5)}]$$

This completes the proof of the lemma.

Now we describe the universal construction for the geometric forms. Fix an infinite sequence of real variables $x_0, x_1, x_2, \dots$ and let A be the algebra of smooth real valued functions in these variables such that each function depends on only finitely many of them. We set:

$$\Lambda^d_{geo} = \begin{cases} A & \text{for} \quad d = 0, 1 \\ 0 & \text{otherwise} . \end{cases}$$

Then we define

$$d : \Lambda^0_{geo} \to \Lambda^1_{geo}$$

$$\text{by} \quad df = \sum_i x_{i+1} \frac{\partial f}{\partial x_i} .$$

Given a specific vertex free curve γ, let $\Lambda^*(\gamma)$ denote the DeRham complex. The map

$$\Psi_\gamma : \Lambda^d_{geo} \to \Lambda^d(\gamma)$$

is defined by:

$$f \mapsto f(Q^{(0)}, Q^{(1)}, \ldots) \qquad \text{for} \quad d = 0$$
$$f \mapsto f(Q^{(0)}, Q^{(1)}, \ldots)\omega \quad \text{for} \quad d = 1 .$$

This map is clearly a chain map. Moreover it is clear that for any form $\eta \in \Lambda^*_{geo}$, the assignment $\gamma \to \Psi_\gamma(\eta)$ is a geometric form. We claim that in fact *every* geometric form arises in this way. Since every geometric 1-form Ω is a multiple of the non-vanishing geometric 1-form ω, we may write $\Omega = R\omega$, where R is a geometric function. Thus it suffices to show that every geometric *function* H is of the form $\gamma \to \Psi_\gamma(\eta)$ for some function $\eta \in \Lambda^0_{geo}$. To see this we first consider the smooth $r - 4$ parameter family of curves P given by the equation

$$y = x^3 + t_5 \, x^5 + t_6 x^6 + \ldots + t_r x^r .$$

Set $t = (t_5, t_6, \ldots t_r)$. These curves are all vertex free at the origin, and by the result of §4 we know that for an arbitrary curve γ and an arbitrary point $p \in \gamma$ on it, (γ, p) has contact of order $\geq r$ with *some* member of this $r - 4$ parameter family of curves. It follows from the invariance property (i) that we need only find $\eta \in \Lambda^0_{geo}$, such that $\Psi_\gamma(\eta) = H_\gamma$ at the origin for all γ in the family P. By the smoothness property (ii) we can write $H_{\gamma(t)}(0) = L(t)$, for some smooth function L, and by the lemma above $t_5, t_6, \ldots t_r \in \mathbf{Q}[Q, Q^{(1)}, \ldots, Q^{(r)}]$. Thus $L(t) = \eta(Q^{(0)}, Q^{(1)}, \ldots, Q^{(r)})$ for some smooth function $\eta \in \Lambda^0_{geo}$, and we are done.

We remark that although Λ^*_{geo} gives all the smooth *local* invariants of curves in $\mathbf{R}^2$, it certainly does not give other, more global, invariants like $\nu = \int \omega$.