1. Sets of bounded height

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III. MORDELL'S CONJECTURE

Suppose L is a field of characteristic zero of finite type over a relatively algebraically closed subfield K.

THEOREM 3.1 (Manin). *Suppose C is a curve of genus at least 2 defined over K. Suppose $C(L)$ is infinite, then there exists a curve C_0 defined over K such that $C_0 \times_K L \cong C$ and $C(K)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

We can translate this into

THEOREM 3.1 (BIS). *Suppose S is a variety defined over K and suppose $C \to S$ is a smooth proper curve of genus at least 2 over S. Suppose $C(S)$ is infinite, then there exists a curve C_0 defined over K such that $C_0 \times_K S \cong C$ and $C(S)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

Remarks. First, it is possible to reduce this by standard arguments to the case in which S is a smooth affine curve over K and so we will suppose this to be the case. Second, if we can prove that $C_0 \times_K X \cong C$ for some C_0 defined over K, (i.e. that C is a constant family) then this is de Franchis' theorem which is proven in Lang's Fundamentals of Diophantine Geometry. Hence to prove this theorem all we have to do is show that if $C(S)$ is infinite then C is a constant family of curves.

1. SETS OF BOUNDED HEIGHT

In this section we will either recall or derive the properties of heights needed in the sequel.

Let $f: X \to S$ be a smooth projective morphism of varieties over K a field of characteristic zero. Corresponding to a projective embedding of X over S, there exists a function $h: X(S) \to \mathbf{R}$ called a logarithmic height. (For a reference, see ([L-FD] Chapter 3, §3). If the logarithmic height of a subset of $X(S)$ is bounded with respect to one projective embedding, it is bounded with respect to all (See [L] Prop. 1.7, Chapt. 4). We will call such a set a set of bounded height and a set of points which is not of bounded height, a set of unbounded height. We will need several properties of such sets. If $g: X' \to X$ is a morphism of projective schemes over S which is finite onto its image, then the inverse image of a set of bounded height in $X(S)$ is a set of bounded height in $X'(S)$. Suppose X is an Abelian scheme over S and R is the subgroup of $X(S)$ consisting of constant sections of X/S. Let $s \in X(S)$. Then the set $s + R$ is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose E is a finite dimensional K vector subspace of $K(C)$. Then the set*

$$T = \{s \in C(S) \colon \exists\, k \neq 0 \in E \quad \text{such that} \quad s^*k = 0\}$$

has bounded height.

Proof. Without loss of generality we may increase E to suppose that the rational map $g\colon C \to \mathbf{P}_K(E)$ given on points by $x \to (e \in E \to e(x))$ is birational onto its image (note: g is actually a morphism on the compliment of the polar locus of E). It follows that g induces an embedding of the generic fiber of C/S into $\mathbf{P}_{K(S)}(E \otimes K(S))$. Let h denote the logarithmic height with respect to this embedding. It follows that if $s \in C(S)$, $g \circ s$ is constant or $g \circ s$ has degree one. In the former case $h(s)$ is zero and the degree of the Zariski closure of $g \circ s(S)$ in $\mathbf{P}(E)$ in the latter.

Now if $s \in T$, and $g \circ s$ is not constant, it follows that the Zariski closure of $g \circ s(S)$ is a component of a hyperplane section of the Zariski closure of $g(C)$. Hence, $h(s)$ is less than or equal to the degree of the Zariski closure of $g(C)$. This proves the lemma. □

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose $C \to S$ is as in the above theorem. If $C(S)$ contains an infinite set of bounded height then C is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of $C(S)$ have bounded height.

2. LANG-SIEGEL TOWERS

Suppose the genus of C is at least 1. Suppose T is an infinite subset of $C(S)$.

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\}), m, n \in \mathbf{Z}_{>0}$ and $n \leqslant m$, over K such that

(i) $C_1 = C$,

(ii) *$h_{m,n}\colon C_m \to C_n$ is étale,*