5. The algebraic proof

Objekttyp: Chapter

Zeitschrift: L'Enseignement Mathématique

Band (Jahr): 36 (1990)

Heft 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

PDF erstellt am: 29.04.2024

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Proof. Let s and t be elements of A(S). Define the map $g: A \to A$ by $g = m \circ (id, t \circ f) (g(x) = x + t(f(x)))$. Then $g^*: H^1_{DR}(A/S) \to H^1_{DR}(A/S)$ is the identity so that $g^*M(e, s) = M(e, s)$ on the one hand and $g^*M(e, s) = M(t, s + t)$ by Proposition 1.3.2 on the other. Hence,

M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)by Proposition 1.3.1.

Let (B, τ) denote the K(S)/K trace of $A_{K(S)}$ (see [L-AV]). In particular, B is an Abelian scheme over K and $\tau: B \times \operatorname{spec}(K(S)) \to A_{K(S)}$ is a homomorphism. Since K has characteristic zero τ is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of $A_{K(S)}$ defined over over K. The morphism τ extends uniquely to an S-morphism $\overline{\tau}: B \times_K S \to A$. It follows that B(K) maps naturally into A(S). We call the elements s of A(S) such that ns is in the image of B(K), the constant sections of A/S.

PROPOSITION 1.4.2. The kernel of M contains all constant sections of A/S.

Proof. Let s be a constant section of A/S. Then there exists a positive integer n such that $ns = \bar{\tau} \circ (t \times id)$ where $t \in B(K)$. Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^*M(t \times id) = 0$. Since

$$\operatorname{Ext}(H^1_{DR}(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows. \Box

We wish to prove the conserve of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. The kernel of M is precisely the group of all constant sections of A/S.

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

5. The Algebraic proof

a. Differentials with logarithmic singularities

(See [K] §1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and

cross normally relative to T. Let W = X - Z and \tilde{Z} the disjoint union of the irreducible components of Z. Let $(\Omega_{X/T}(\text{Log}(Z)), d)$ denote the complex of differentials on X/T with logarithmic singularities along Z. (When T = K, we drop T from the notation.) When T has characteristic zero, which we will now assume, the *i*-th hypercohomology group of this complex is naturally isomorphic to $H_{DR}^i(W/T)$. We have a natural short exact sequence of complexes

$$0 \to \Omega^{:}_{X/T} \to \Omega^{:}_{X/T}(\operatorname{Log}(Z)) \xrightarrow{\operatorname{Res}} \Omega^{:}_{Z/T}(-1) \to 0$$

From which, upon taking cohomology, we obtain the long exact sequence:

(5.1)
$$0 \to H^1_{DR}(X/T) \to H^1_{DR}(W/T) \to H^0_{DR}(\tilde{Z}/T) \to H^2_{DR}(X/T)$$
$$\to H^2_{DR}(W/T) \to H^2_{DR}(\tilde{Z}/T)$$

In addition, we have a short exact sequence of complexes

 $0 \to \Omega^1_S \otimes \Omega^{\boldsymbol{\cdot}}_{X/S}(\operatorname{Log}(Z))(-1) \to \Omega^{\boldsymbol{\cdot}}_X(\operatorname{Log}(Z)) \to \Omega^{\boldsymbol{\cdot}}_{X/S}(\operatorname{Log}(Z)) \to 0 \ .$

The boundary maps in the long exact sequence of hypercohomology obtained from this short exact sequence are the Gauss-Manin connections $\nabla: H^i_{DR}(W/S) \to \Omega^1_S \otimes H^i_{DR}(W/S)$. Moreover the long exact sequence (5.1) is horizontal with respect to all the Gauss-Manin connections.

If D is any divisor on X, let $\eta_T(D)$ denote the cohomology class of D in $H_{DR}^2(X/T)$. Recall ([H-DR; 7.7]), if \mathcal{L} is an ordered affine open cover of C and $\{f_U\}$ is a Čech one-cochain with with coefficients in \mathcal{P}_A with respect to \mathcal{L} such that the divisor of f_U is the restriction of D to U, then $\eta_T(D)$ is the cohomology class represented by the hyper one-cocycle $(0, \{d_{C/T} \text{Log}(f_{U,V})\}, 0)$, where $f_{U,V} = f_U/f_V(U < V)$. Suppose now that T is affine. Then $H_{DR}^0(\tilde{Z}/T)$ is naturally isomorphic to the group of divisors on X supported on Z with coefficients in K[T].

LEMMA 1.5.1. Suppose D is a divisor on X supported on Z, then the image of D in $H^2_{DR}(X/T)$ via the appropriate map in (5.1) is equal to $\eta_T(D)$.

Proof. This is essentially Proposition 7.6 of [H]. We carry out the proof in order to "straighten out" the sign.

Let \mathscr{L} be an affine open cover of X and $\{f_U\}$ is a Čech one-chain with coefficients in \mathscr{D}_X with respect to \mathscr{L} such that the divisor of f_U is the restriction of D to U. Then, $d\operatorname{Log}(f_U) \in \Omega^1_{X/T}(\operatorname{Log}(Z))(U)$ and $\operatorname{Res}(d\operatorname{Log}(f_U))$ is the image of the image of D in $H^0_{DR}(\tilde{Z}/T) \cong \mathscr{D}_{\tilde{Z}}(U)$. It

follows that the image of D in $H^2_{DR}(X/T)$ is the class of the hypercoboundary of $(\{d \operatorname{Log}(f_U)\}, 0)$ which is $\eta_T(D)$ by definition.

By a properly semi-stable curve over S, we mean a curve over S such that the irreducible components of the closed fibers are smooth and cross normally. (The irreducible components do not have to be smooth if the curve is only semistable.)

COROLLARY 1.5.2. Suppose R is a smooth connected curve over a field K and X is a properly semi-stable curve over R smooth over K. Suppose U is a non-empty open subset of R and Y = R - U. Then the kernel of the natural map from $H^2_{DR}(X)$ into $H^2_{DR}(X_U)$ is generated by $\{\eta(D)\}$ where D runs over the irreducible components of X_Y .

Proof. This follows from the lemma and the exact sequence (5.1), since the closed fibers of C/T are unions of smooth hypersurfaces of C which cross normally. \Box

LEMMA 1.5.3. With notation as in the above corollary, if R is affine and X is smooth over R then the map from $H^2_{DR}(X) \rightarrow H^2_{DR}(X_U)$ is an injection.

Proof. For a closed point x of R, let X_x denote the fiber above x. Since all the fibers of X over S are smooth, it follow from the corollary that the the kernel of the map $H^2_{DR}(X) \to H^2_{DR}(X_U)$ is generated by $\{\eta(X_x)\}$ where x runs over the closed points of Y. Now $\eta(X_x)$ is the pull-back of $\eta(x) \in H^2_{DR}(R)$. As this latter group is zero, this proves the lemma. \square

b. End of algebraic proof

First by using the functoriality of M, Proposition 1.3.2, and the fact that every Abelian variety over S is the quotient of a Jacobian over S we may assume that A is the Jacobian of a smooth proper curve C over S. By Proposition 1.1.1 and the long exact sequence (2.3), $\text{Ext}(H_{DR}^1(C/S)^{\vee}, K[S])$ maps naturally into $H_{DR}^2(C)$. Moreover, since C is a proper smooth connected curve over $S, H_{DR}^1(C/S)$ is canonically isomorphic to $H_{DR}^1(C/S)^{\vee}$. The fact we need to finish the proof is:

PROPOSITION 1.5.4. Let s and t be two elements of C(S). The class $\eta(t-s)$ is equal to the image of M(s,t) in $H^2_{DR}(C)$.

By the previous lemma and the functoriality of η we may shrink S to suppose that $s \cap t = \emptyset$. To prove the proposition, we need the next lemma.

Suppose now that $T = S, Z = s \cup t$ and X = C. Then the exact sequence (5.1) becomes:

(5.2) $0 \to H^1_{DR}(C/S) \to H^1_{DR}(W/S) \to H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S) \to 0$

Furthermore $H_{DR}^2(C/S)$ is canonically isomorphic to K[S] with generator $\eta_S(s) = \eta_S(t)$ and so the kernel of $H_{DR}^0(\tilde{Z}/S) \to H_{DR}^2(C/S)$ is a principal K[S] module with generator D = s - t. Using this generator, (5.2) yields an extension $B_{s,t}$ of the connection (K[S], d) by $(H_{DR}^1(C/S), \nabla)$.

LEMMA 1.5.5. Identifying $H^1_{DR}(C/S)$ with $H^1_{DR}(C/S)^{\vee}$, the extension $B_{s,t}$ is isomorphic to the dual of $E_{s,t}$.

Proof. Regarding the complexes $\Omega_{C/S,Z}$ and $\Omega_{C/S}(\text{Log}(Z))$ as subcomplexes of $\Omega_{W/S,Z}$ the wedge product gives a product from

$$\Omega^{\cdot}_{C/S,Z} \times \Omega^{\cdot}_{C/S}(\operatorname{Log}(Z))$$

into $\Omega_{C/S}^{\cdot}$ which induces a pairing

$$(,): H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S) \to H^2_{DR}(C/S) \cong K[S]$$
.

This pairing is compatible with the exact sequences

$$0 \to H^0(C, \Omega^1_{C/S}) \to H^1_{DR}(C/S, Z) \to H^1(C, \Omega^0_{X/S}) \to 0$$
$$0 \to H^0(C, \Omega^1_{C/S}(\operatorname{Log}(Z)) \to H^1_{DR}(W/S) \to H^1(C, \mathscr{D}_C) \to 0$$

arising from the Hodge to de Rham spectral sequences for hypercohomology (which degenerate). In other words, the image of $H^0(C, \Omega^1_{C/S})$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$ in $H^1_{DR}(W/S)$ and if we identify $\Omega^0_{X,Z}$ with $\mathscr{P}_C(Z)$ and $\Omega^1_{C/S}(\text{Log}(Z))$ with $\Omega^1_{C/S}(-Z)$ the pairings induced on $H^0(C, \Omega^1_{C/S}) \times H^1(C, \mathscr{P}_C)$ and on

$$H^1(C, \Omega^0_{X,Z}) \times H^0(C, \Omega^1_{C/S}(\operatorname{Log}(Z)))$$

are the natural ones. Since these pairings are non-degenerate, it follows that the pairing on $H_{DR}^1(C/S, Z) \times H_{DR}^1(W/S)$ is non-degenerate.

It is also clear that the image of $H^0_{DR}(Z/S) \cong K[Z]$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^1_{DR}(C/S)$ in $H^1_{DR}(W/S)$ and that the pairing induced on $H^1_{DR}(C/S) \times H^1_{DR}(C/S)$ is the natural one.

The lemma will follow from the following claim: Let ι denote the map from K[Z] to $H^1_{DR}(C/S, Z)$ and Res the map from $H^1_{DR}(W/S)$ to K[Z]. Let $T_{Z/S}$ denote the trace from K[Z] to K[S]. Suppose $c \in K[Z]$ and $\omega \in H^1_{DR}(W/S)$. Then

$$(\iota(c), w) = - T_{Z/S}(c \operatorname{Res}(\omega))$$

Indeed, if $s^*c = 0$, $t^*c = 1$, $\operatorname{Res}_s(\omega) = 1$ and $\operatorname{Res}_t(\omega) = -1$ then $-T_{Z/S}(c\operatorname{Res}(\omega)) = 1$.

To prove this claim we may shrink S. Hence, we may assume first that $\{U, V\}(U < V)$ is an ordered affine open cover of C such that U = C - s and V = C - t, second that $\iota(c)$ is represented by a hypercocycle of the form $\partial(\{g_U, g_V\})$ where $s^*g_U = s^*c$ and $t^*g_V = t^*c$ and third, since the composition $H^0(C, \Omega^1_{C/S}(\text{Log}(Z)) \to H^1_{DR}(W/S) \to K[Z]$ is surjective, that w is in the image of $H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$, i.e., w is represented by a hypercocycle of the form $(\{\omega_U, \omega_V\}, 0\}$ where $\omega_U = \omega = \omega_V$ on $U \cap V$ for some $\omega \in H^0(C, \Omega^1_{C/S}(\text{Log}(Z)))$. It follows that $(\iota(c), w)$ as an element of $H^1_{DR}(C, \Omega^1_{C/S}) \cong H^2_{DR}(C/S)$ is represented by the cocycle $\{v_{U,V}\}$ with $v_{U,V} = (g_V - g_U)\omega$. Since the image of this element in K[S] is

$$\operatorname{Res}_{s}(-g_{U}\omega) + -\operatorname{Res}_{t}(g_{V}\omega) = -(s^{*}g_{U}\operatorname{Res}_{s}(\omega) + t^{*}g_{V}\operatorname{Res}_{t}(\omega))$$
$$= -T_{T/S}(c\operatorname{Res}(\omega)).$$

this establishes the claim and the lemma. \Box

End of proof of Proposition 1.5.4

Consider the commutative diagram of complexes of sheaves with exact rows and columns

If we take hyper-cohomology of this diagram we obtain a commutative diagram

$$H^{1}_{DR}(C/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(C/S)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{1}_{DR}(W/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(W/S)$$

$$\downarrow$$

$$H^{0}_{DR}(Z) \rightarrow H^{0}_{DR}(Z/S)$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{1}_{DR}(C/S) \xrightarrow{\nabla} \Omega^{1}_{S} \otimes H^{1}_{DR}(C/S) \rightarrow H^{2}_{DR}(C) \rightarrow H^{2}_{DR}(C/S)$$

with exact rows and columns in which the bottom row is part of the Leray long exact sequence. Let *a* be the element in $H_{DR}^0(Z)$ corresponding to the divisor s - t. The image of *a* in $H_{DR}^2(C)$ is $\eta(s - t)$ by Lemma 1.5.1. On the other hand the image of *a* in $H_{DR}^0(Z/S)$ is our chosen generator of the kernel of the map to $H_{DR}^2(C/S)$. In particular, it is the image of an element *b* of $H_{DR}^1(W/S)$ and $\nabla(b)$ is the image of an element *c* of $\Omega_s^1 \otimes H_{DR}^1(C/S)$ whose image in $H_{DR}^2(C/S)$ is the same as that of *a* by an elementary diagram chase. On the other hand, the image of *c* in $H^1(H_{DR}^1(C/S), \nabla)$ is the class corresponding to the extension $B_{s,t}$ by definition (see Proposition 1.1.1) which is, after identifying $H_{DR}^1(C/S)$ with $H_{DR}^1(C/S)^{\vee}$, -M(s,t) by Lemma 1.5.1 and Lemma 1.5.3. Hence the image of M(s,t) in in $H_{DR}^2(C)$ is $-\eta(s-t) = \eta(t-s)$ as required. \Box

Now we are in a position to prove the Theorem 1.4.3. We will suppose M(s, t) = 0 which amounts to $\eta_S(t - s) = 0$ by the Proposition 1.5.1. Recall, that A is the Jacobian of C/S. Let d denote the divisor class of t - s in A(K(C)). We will show that the canonical height of d is zero. We may replace S by a finite étale cover and complete C to a properly semi-stable curve \tilde{C} over the completion \tilde{S} of S which is smooth over K. Let D be a **Q**-rational divisor on \tilde{C} which is perpendicular (under the intersection pairing) to all the irreducible components of all the fibers of \tilde{C}/\tilde{S} and whose restriction to C is t - s. Such a divisor exists by the function field analogue of Theorem 1.3 of [Hr] (see also Theorem 5.1 (i) of [Ch]). It follows that the image of $\eta(D)$ in $H^2_{DR}(C)$ is $\eta(t - s) = 0$. Corollary 1.5.2 implies that $\eta(D)$ is in the span of $\{\eta(Y)\}$ where Y runs over the irreducible component of the closed fibers above $\tilde{C} - C$. In particular, $D \cdot D = 0$ using Theorem 7.8.2 of [H]. On the other hand, $D \cdot D$ is -2 times the canonical height of d by the function field analogue of Theorem 5.1 of [Ch]. It now follows from Theorem 5.4.1 of [L],

that the image of t - s in J(C) is a constant section which completes the proof. \Box

6. The analytic proof

In this section we will suppose $K = \mathbf{C}$.

a. The Poincaré Lemma

Suppose (\mathcal{S}, ∇) is a sheaf on S^{an} with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathscr{S} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \Omega^2_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf $\mathscr{S}^{\bigtriangledown}$. Hence,

PROPOSITION 1.6.1. $H^{i}(\mathcal{S}, \nabla)$ is naturally isomorphic to $H^{i}(S, \mathcal{S}^{\nabla})$.

Remark. As in Proposition 1.1.1, $H^1(\mathcal{G}, \nabla)$ is isomorphic to $Ext(\mathcal{G}^{\vee}, \mathcal{O}_{San})$. We can describe the isomorphism from $H^1(\mathcal{G}, \nabla)$ to $H^1(S, \mathcal{G}^{\nabla})$ explicitly as follows: Let *h* be an element of $H^1(\mathcal{G}, \nabla)$. Let \mathcal{G} is a covering of *S* by open disks. Suppose \mathcal{G} is an extension of \mathcal{G}^{\vee} by \mathcal{O}_{San} corresponding to *h*. Then \mathcal{G}^{\vee} is an extension of \mathcal{O}_{San} by \mathcal{G} . For each $U \in \mathcal{G}$, there exists an $s_U \in \mathcal{G}^{\vee}(U)^{\nabla}$ which maps to 1 in $\mathcal{O}_{San}(U)$. Then the image *h* in $H^1(S, \mathcal{G}^{\nabla})$ is the class of the cocycle $\{(U, V) \to s_U - s_V\}$.

Suppose, X is a smooth proper S-scheme and Z is a subscheme of X which is either empty or finite over S. We will define the Betti homology sheaf $\mathscr{M}_1(X/S, Z, \mathbb{Z})$ on S^{an} as follows. If Z is smooth over S, we define $\mathscr{M}_i(X/S, Z, \mathbb{Z})$ to be the sheaf associated to the presheaf

$$U \rightarrow H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbb{Z})$$
,

(this latter group is the Betti homology of $f^{-1}(U)$ relative to $f^{-1}(U) \cap Z$). More generally, let S' be a non-empty affine open subset of S such that $Z' = Z \times_S S'$ is étale over S'. Let $X' = X \times_S S'$ and let ι denote the inclusion morphisms $X' \to X, Z' \to Z$ and $S' \to S$. We set

$$\mathscr{M}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathscr{M}_i(X'/S', Z', \mathbf{Z})$$

This is independent of the choice of S'. We also set

 $\mathcal{M}_i(X/S, \mathbb{Z}) = \mathcal{M}_i(X/S, \mathcal{O}, \mathbb{Z}) \text{ and } \mathcal{M}_1(X/S, \mathbb{Z}, \mathbb{C}) = \mathcal{M}_1(X/S, \mathbb{Z}, \mathbb{Z}) \otimes \mathbb{C}$.