6. Periods of reduced cycles

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$< \dfrac{2a_{n_0-1}}{a_{n_0}}$. Then, appealing to (5.20), we obtain

$$1 < \phi_1 \dots \phi_{n_0} < \frac{2a_0}{a_{n_0} B_{n_0-3}},$$

so that, by (5.17), we have

$$\frac{a_{n_0}}{a_0} < \delta < \frac{2}{B_{n_0-3}}.$$

It remains to consider the case $n_0 = 1$. If I_0 is reduced then $\delta = 1$. If I_0 is not reduced then $\delta = \dfrac{a_1}{a_0}\phi_1$ and, as above, we have $1 < \phi_1 < \dfrac{2a_0}{a_1}$, giving $\dfrac{a_1}{a_0} < \delta < 2$.

Hence in all cases we have $\dfrac{1}{a_0} \leqslant \delta < 2$. All subsequent Lagrange neighbours of I are reduced by Lemma 5. This completes the proof of Proposition 7.

6. Periods of reduced cycles

We show that any two equivalent reduced, primitive ideals of the same order O_D can be obtained from one another by using the Lagrange reduction process described in §5.

PROPOSITION 8. ([5]: §31, [12]: Theorem 4.5) *Let* $I = a[1, \phi]$ $(a > 0)$ *and* $J = b[1, \psi]$ $(b > 0)$ *be two equivalent, reduced, primitive ideals of* O_D, *so that* $[1, \psi] = \rho[1, \phi]$ *for some* $\rho(>0) \in K^*$. *Interchanging* I *and* J *if necessary we may suppose that* $\rho \geqslant 1$. *Set* $I_0 = I$. *Then there exists a non negative integer* n *such that* $J = I_n$ *and* $\rho = \phi_1 \dots \phi_n$, *so that* $J = I_n = \rho_n I$.

Proof. Recalling that $\phi_n > 1 \, (n \geqslant 1)$, we see from (5.10) and (5.13) that the sequence $\{\phi_1 \dots \phi_n\}_{n=0}^{\infty}$ is monotonically increasing and unbounded. Hence there exists an integer $n \geqslant 0$ such that $\phi_1 \dots \phi_n \leqslant \rho < \phi_1 \dots \phi_{n+1}$. As $I_n = \dfrac{a_n}{a_0}\phi_1 \dots \phi_n I_0$ (by (5.5)), we have $\dfrac{1}{b}J = \dfrac{\rho}{\phi_1 \dots \phi_n}\dfrac{1}{a_n}I_n$. If $\rho = \phi_1 \dots \phi_n$ then

$\frac{1}{b} J = \frac{1}{a_n} I_n$ and so, by Proposition 2 (iii), we have $b = a_n$ and $J = I_n$ as required. This we may suppose that $\rho > \phi_1 \dots \phi_n$. Replacing I_0 by I_n, we obtain

$$\text{(6.1)} \qquad \frac{1}{b} J = \rho \frac{1}{a_0} I_0, \quad \text{where} \quad 1 < \rho < \phi_1 .$$

From (6.1), we see that $\frac{a_0}{\rho} J = bI_0$, and so, as $J\bar{J} = (b)$, we have $\frac{a_0}{\rho} = I_0\bar{J}$, showing that $\frac{1}{\rho} \in \frac{1}{a_0} I_0$. Next we observe that

$$\frac{1}{a_0} I_0 = \frac{1}{\phi_1 a_1} I_1 = \frac{1}{\phi_1} [1, \phi_1] = \left[1, \frac{1}{\phi_1}\right] ,$$

so there are integers x and y such that

$$\frac{1}{\rho} = x + \frac{y}{\phi_1} .$$

Thus, as $1 < \rho < \phi_1$, we have

$$\text{(6.2)} \qquad \frac{1}{\phi_1} < x + \frac{y}{\phi_1} < 1 .$$

Appealing to (6.1), we obtain

$$J = \frac{b\rho}{a_0} I_0 = \frac{b\rho}{a_1 \phi_1} I_1 = \frac{b\rho}{\phi_1} [1, \phi_1] ,$$

so that $\frac{b\rho}{\phi_1} \in J$, and $0 < \frac{b\rho}{\phi_1} < b$. As J is reduced, by Proposition 4, we have $\left| \frac{b\bar{\rho}}{\bar{\phi}_1} \right| = \frac{b\,|\,\bar{\rho}\,|}{|\,\bar{\phi}_1\,|} > b$, so that $\left| \frac{1}{\bar{\rho}} \right| < \left| \frac{1}{\bar{\phi}_1} \right|$, that is

$$\text{(6.3)} \qquad \left| x + \frac{y}{\bar{\phi}_1} \right| < \frac{1}{|\,\bar{\phi}_1\,|} .$$

From (6.2) we see that $y \neq 0$. Then (6.3) shows that $x \neq 0$, and that, as $\bar{\phi}_1 < 0, xy > 0$. This contradicts (6.2), and completes the proof of Proposition 8.

Let I_0 be a reduced, primitive ideal of a class C of O_D. By the Lagrange reduction process described in §5, we obtain (by Proposition 5) an infinite sequence $\{I_n\}_{n=0}^{\infty}$ of reduced, primitive ideals with each ideal I_n equivalent to I_0. By Proposition 8, this sequence contains all the reduced, primitive ideals of the class C. As C contains only a finite number of reduced, primitive ideals (§4), there exist integers r and l with $0 \leqslant r < r + l$ such that $I_r = I_{r+l}$. Applying Proposition 6 (ii), we obtain successively $I_{r-1} = I_{r+l-1}, I_{r-2} = I_{r+l-2}, \ldots$, and, after r steps, we have $I_0 = I_l$, which shows that the sequence $\{I_n\}_{n=0}^{\infty}$ is purely periodic.

Definition 12. (Period) Let I_0 be a reduced, primitive ideal of a class C of O_D. Let l be the least positive integer with $I_0 = I_l$. The set $\{I_0, \ldots, I_{l-1}\}$ is called the *period* of the class C. The length of the period is the integer l.

The period of the class C of O_D consists of all the reduced, primitive ideals in C. It is easy to see that if $I_s = I_t$ then l divides $s - t$. As $I_l = I_0$, we see, from (5.5), that $I_0 = \eta I_0$, where

$$\eta = \rho_l = \prod_{i=1}^{l} \phi_i \,, \tag{6.4}$$

and so, by Proposition 2 (ii), η is a unit (> 1) of O_D.

PROPOSITION 9. (i) *If $I = I_0$ and J are equivalent, reduced, primitive ideals of O_D with $J = \alpha I_0$, where $\alpha (\geqslant 1) \in K^*$, then there exist unique integers q and s such that*

$$\alpha = \eta^q \rho_s \quad (\rho_s \text{ is defined in (5.5), } \eta \text{ in (6.4)})$$

where

$$q \geqslant 0 \,, \quad 0 \leqslant s \leqslant l - 1 \,.$$

(ii) If $J = I$ then we have $s = 0$ and $\alpha = \eta^q$.

Proof. (i) By Proposition 8 there exists a nonnegative integer n such that

$$J = I_n = \rho_n I_0 \,, \quad \alpha = \rho_n \,.$$

Let $q (\geqslant 0)$ and s be the integers defined uniquely by

$$n = ql + s \,, \quad 0 \leqslant s \leqslant l - 1 \,.$$

Then, by periodicity, we have

$$\alpha = \rho_s (\rho_l)^q = \eta^q \rho_s \,,$$

where

$$\eta = \rho_l = \phi_1 \ldots \phi_l .$$

This shows the existence of the integers $q(\geqslant 0)$ and $s(0 \leqslant s \leqslant l-1)$.

We next show that q and s are unique. Suppose we have $\alpha = \eta^{q_1}\rho_{s_1} = \eta^{q_2}\rho_{s_2}$ with $s_1 \leqslant s_2$. If $s_2 > s_1$ then $q_1 > q_2$ and, appealing to (5.5) and recalling that $-1 < \bar{\phi}_i < 0 (i \geqslant 1)$, we obtain

$$\eta \leqslant \eta^{q_1 - q_2} = \frac{\rho_{s_2}}{\rho_{s_1}} = \prod_{i=s_1+1}^{s_2} \left(\frac{-1}{\bar{\phi}_i}\right) < \prod_{i=1}^{l} \left(\frac{-1}{\bar{\phi}_i}\right) = \eta ,$$

which is a contradiction. Hence we must have $s_1 = s_2$. Then $\eta^{q_1} = \eta^{q_2}$ and, as $\eta > 1$, we must have $q_1 = q_2$. This completes the proof of (i).

(ii) From the proof of (i) we see that $I_n = J = I_0$, so that $l \mid n$, and thus $q = n/l$ and $s = 0$.

COROLLARY 5. *$\eta = \prod_{i=1}^{l} \phi_i$ is a unit (>1) of O_D such that every unit ε of O_D is given by $\varepsilon = \pm \eta^r$, where r is an integer. η is called the fundamental unit of O_D.*

Proof. Let ε be a unit of O_D and let

$$\delta = \begin{cases} \varepsilon , & \text{if} \quad \varepsilon \geqslant 1 , \\ 1/\varepsilon , & \text{if} \quad 0 < \varepsilon < 1 , \\ -1/\varepsilon , & \text{if} \quad -1 < \varepsilon < 0 , \\ -\varepsilon , & \text{if} \quad \varepsilon \leqslant -1 , \end{cases}$$

so that δ is a unit of O_D satisfying $\delta \geqslant 1$. Applying Proposition 9 (ii) to I_0 and $J = \delta I_0$, we see that $\delta = \eta^q$, and so $\varepsilon = \pm \eta^r$.

Corollary 5 was first proved by Lagrange in the case of the principal class [3: p. 452] (see also [8]). We see that the theory of periods of reduced, primitive ideals in O_D not only gives the structure of the group of units of O_D but also provides the structure of each period (the "infrastructure" of Shanks [7]).

COROLLARY 6. *With I_0 a reduced, primitive ideal of O_D, we have*

(i) $\eta = B_{l-1}\phi_0 + B_{l-2}$,

(ii) $\eta = A_{l-1} - B_{l-1}\bar{\phi}_0$,

(iii) $l \log \left(\frac{1+\sqrt{5}}{2}\right) \leqslant \log \eta < l \log \sqrt{D}$

Proof. Taking $n = Nl (N = 1, 2, \ldots)$ in (5.13) we obtain, as $\phi_{Nl} = \phi_0$,

$$\eta^N = B_{Nl-1}\phi_0 + B_{Nl-2} . \tag{6.5}$$

The assertion (i) is the case $N = 1$.

From (5.7), (5.9) and (5.13), we obtain for $n \geqslant 1$

$$\phi_1 \ldots \phi_n = \frac{(-1)^{n-1}}{B_{n-1}\phi_0 - A_{n-1}} .$$

Taking $n = Nl (N = 1, 2, \ldots)$ and recalling that $\eta\bar{\eta} = (-1)^l$, we obtain $\eta^N = -\dfrac{(\eta\bar{\eta})^N}{B_{Nl-1}\phi_0 - A_{Nl-1}}$, so that taking conjugates we deduce

$$\eta^N = A_{Nl-1} - B_{Nl-1}\bar{\phi}_0 . \tag{6.6}$$

The assertion (ii) is the case $N = 1$.

From (6.5) and (5.10) we have

$$\eta^N > B_{Nl-1} + B_{Nl-2} \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-3} = \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-1},$$

so that

$$\eta > \left(\frac{1+\sqrt{5}}{2}\right)^{l-(1/N)} \qquad (N = 1, 2, 3, \ldots) .$$

Letting $N \to \infty$, we obtain

$$\eta \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^l ,$$

proving the first equality in (iii).

Finally, as $\phi_i < \sqrt{D} (i \geqslant 0)$, we have

$$\eta = \phi_1 \ldots \phi_l < (\sqrt{D})^l ,$$

proving the second assertion in (iii).

Example 3. $(D = 1892)$ The period of the class containing the ideal $[1, 21 + \sqrt{473}]$ is

$$\{[1, 21 + \sqrt{473}],\ [32, 21 + \sqrt{473}],\ [11, 11 + \sqrt{473}],\ [32, 11 + \sqrt{473}]\} .$$

Thus, by Corollary 5, the fundamental unit of O_{1892} is

$$(21 + \sqrt{473}) \left(\frac{21+\sqrt{473}}{32}\right) \left(\frac{11+\sqrt{473}}{11}\right) \left(\frac{11+\sqrt{473}}{32}\right)$$

$$= \frac{1}{11.32^2}(21+\sqrt{473})^2(11+\sqrt{473})^2$$
$$= \frac{1}{11.32^2}(704+32\sqrt{473})^2$$
$$= \frac{1}{11}(22+\sqrt{473})^2$$
$$= 87 + 4\sqrt{473}$$
$$= 87 + 2\sqrt{1892}.$$

The period of the class containing the ideal $[7, 16+\sqrt{473}]$ is

$$\{[7,16+\sqrt{473}], [16,19+\sqrt{473}], [19,13+\sqrt{473}], [23,6+\sqrt{473}],$$
$$[8,17+\sqrt{473}], [31,15+\sqrt{473}]\}$$

so, by Corollary 5, the fundamental unit of O_{1892} is also given by

$$\left(\frac{16+\sqrt{473}}{7}\right)\left(\frac{19+\sqrt{473}}{16}\right)\left(\frac{13+\sqrt{473}}{19}\right)\left(\frac{6+\sqrt{473}}{23}\right)\left(\frac{17+\sqrt{473}}{8}\right)\left(\frac{15+\sqrt{473}}{31}\right)$$
$$= \left(\frac{111+5\sqrt{473}}{16}\right)\left(\frac{29+\sqrt{473}}{23}\right)\left(\frac{91+4\sqrt{473}}{31}\right)$$
$$= \frac{(349+16\sqrt{473})}{23}\,\frac{(91+4\sqrt{473})}{31}$$
$$= 87 + 4\sqrt{473} = 87 + 2\sqrt{1892}.$$

We are now in a position to define the distance between two reduced, primitive ideals in the same period.

Definition 13. (Distance between ideals) If I and J are equivalent, reduced, primitive ideals of O_D then we define the (mutiplicative) *distance* $d(I, J)$ from I to J by

$$d(I, J) \equiv \rho_s (\mathrm{mod}^{\times}\ \eta)$$

where ρ_s is given as in Proposition 9 (i).

It is clear that $d(I, I) = 1$.

Example 4. $(D = 1892)$ The two reduced, primitive ideals

$$I = [19, 6+\sqrt{473}] \quad \text{and} \quad J = [31, 16+\sqrt{473}]$$

of O_{1892} are equivalent. Applying the Lagrange reduction process to $[19, 6+\sqrt{473}]$, we obtain

$$[19, 6+\sqrt{473}] \xrightarrow{L} [16, 13+\sqrt{473}] \xrightarrow{L} [7, 19+\sqrt{473}] \xrightarrow{L} [31, 16+\sqrt{473}] \ ,$$

so that

$$\begin{aligned} d(I, J) = \rho_3 &= \frac{31}{19}\left(\frac{13+\sqrt{473}}{16}\right)\left(\frac{19+\sqrt{473}}{7}\right)\left(\frac{16+\sqrt{473}}{31}\right) \\ &= \frac{(13+\sqrt{473})\ (111+5\sqrt{473})}{19\times 16} \\ &= \frac{238+11\sqrt{473}}{19} \ . \end{aligned}$$

On the other hand, applying the Lagrange reduction process to $[31, 16+\sqrt{473}]$, we obtain

$$[31, 16+\sqrt{473}] \xrightarrow{L} [8, 15+\sqrt{473}] \xrightarrow{L} [23, 17+\sqrt{473}] \xrightarrow{L} [19, 6+\sqrt{473}] \ ,$$

so that

$$\begin{aligned} d(J, I) &= \frac{19}{31}\left(\frac{15+\sqrt{473}}{8}\right)\left(\frac{17+\sqrt{473}}{23}\right)\left(\frac{6+\sqrt{473}}{19}\right) \\ &= \frac{(91+4\sqrt{473})\ (6+\sqrt{473})}{31\times 23} \\ &= \frac{2438+115\sqrt{473}}{31\times 23} \\ &= \frac{106+5\sqrt{473}}{31} \ . \end{aligned}$$

We note that

$$\begin{aligned} &\left(\frac{238+11\sqrt{473}}{19}\right)\left(\frac{106+5\sqrt{473}}{31}\right) \\ &= \frac{51243+2356\sqrt{473}}{589} \\ &= 87+4\sqrt{473} = \eta \\ &\equiv 1 \,(\bmod^{\times} \eta) \ . \end{aligned}$$

PROPOSITION 10. *If I and J are equivalent, reduced, primitive ideals of O_D then*

$$d(J,I) \equiv d(I,J)^{-1} \quad (\bmod^{\times} \eta) .$$

Proof. As I and J are in the same period we have $J = \rho I (\rho \in K^*)$ and $I = \sigma J (\sigma \in K^*)$. As $I = \rho^{-1} J$ we have $\sigma \equiv \rho^{-1} (\bmod^{\times} \eta)$, which proves Proposition 10.

7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS

Let C be a primitive class of the order O_{Df^2} and let $\theta(C)$ be the image of C by the mapping θ defined in § 3. As an application of the concept of distance described in § 6, we explain how to define a mapping of the period of C into the period of $\theta(C)$, which approximately preserves distance.

THEOREM 2. *For $D' = Df^2$ let $C \in C_{D'}$ and $\theta(C)$ its image by the surjective homomorphism $\theta: C_{D'} \to C_D$.*

(i) There exists a mapping τ from the period of C into the period of $\theta(C)$ such that for I and I' in the period of C we have, for a choice of d modulo units,

$$\frac{d(I,I')}{8f^7D^{3/2}} < d(\tau(I), \tau(I')) < 8f^7D^{3/2}d(I,I') . \tag{7.1}$$

(ii) When $f = p$ (prime) there exists a mapping σ from the period of C into the period of $\theta(C)$ such that for I and I' in the period of C we have, for a choice d modulo units,

$$\frac{d(I,I')}{2Dp^2} < d(\sigma(I), \sigma(I')) < 2Dp^2 d(I,I') . \tag{7.2}$$

Proof. Let $I = a[1, \phi] (a > 0)$ and $I' = a'[1, \phi'] (a' > 0)$ be two equivalent, reduced, primitive ideals of a class C of $O_{D'} (D' = Df^2)$ with $\phi = \dfrac{b + \sqrt{D'}}{2a}$ and $\phi' = \dfrac{b' + \sqrt{D'}}{2a'}$ reduced. Let $\delta \in K^*$ be such that $I' = \delta I, \delta > 0$.

(i) If $GCD(a, f) = 1$ we set $I_1 = I$. If $GCD(a, f) > 1$, from the proof of Lemma 2, we see that there exists an ideal $I_1 = a_1[1, \phi_1] = \rho I$ in C with