4. COHN-VOSSEN'S THEOREM AND SPACES WITHOUT CONJUGATE POINTS

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γ_t, for each t, to $\bar{p}$. This gives a continuous curve starting at $\bar{p}$ and lying over β, hence coinciding with $\bar{\beta}$. If α and β have the same righthand endpoint, then γ_1 is constant, hence $\bar{\alpha}$ and $\bar{\beta}$ also have the same righthand endpoint.

From here it is straightforward to check that the preimage of $B(p, \varepsilon)$ has the desired form. For instance, the fact that $B(\bar{p}_1, \varepsilon)$ and $B(\bar{p}_2, \varepsilon)$ are disjoint for distinct $\bar{p}_1$, $\bar{p}_2$ in the preimage of p has almost the same proof as above. □

Proof of Theorem 4. Note that $\mathbf{G}_m$ is contractible, hence connected and simply connected. By Lemma 1, it suffices to define a complete interior metric $\mathbf{d}^*$ on $\mathbf{G}_m$, with respect to which the endpoint map is a local isometry and whose topology agrees with that of $\mathbf{d}$. (In general $\mathbf{d}$ is not interior; for example, take M to be a Euclidean circle.) Let $\mathbf{d}^*$ be the interior metric induced by $\mathbf{d}$; that is, let $\mathbf{d}^*(\gamma, \sigma)$ be the infimum of lengths of curves in $(\mathbf{G}_m, \mathbf{d})$ from γ to σ. Since these lengths are greater than or equal to $\mathbf{d}(\gamma, \sigma)$, we have $\mathbf{d} \leqslant \mathbf{d}^*$. By Theorem 2, the endpoint map is a local isometry from $(\mathbf{G}_m, \mathbf{d})$ onto M. It follows by the definition of $\mathbf{d}^*$ that every element of $\mathbf{G}_m$ has a neighborhood on which $\mathbf{d}$ and $\mathbf{d}^*$ coincide. It only remains to verify that $\mathbf{d}^*$ is complete; but since $\mathbf{d}$ is complete and $\mathbf{d} \leqslant \mathbf{d}^*$, any $\mathbf{d}^*$-Cauchy sequence converges in $\mathbf{d}$ and hence in $\mathbf{d}^*$. □

4. Cohn-Vossen's theorem and spaces without conjugate points

We have seen that in complete locally convex spaces, the endpoint map on $(\mathbf{G}_m, \mathbf{d})$, which we may denote by $\exp_m$, is a covering map. Such an argument will be more difficult to make if we merely assume that our spaces have no conjugate points; in fact, we have only been successful under the additional assumption of local compactness. Recall that the Hopf-Rinow theorem is used to prove the corresponding theorem in Riemannian geometry. To follow this lead would require a very general version of the Hopf-Rinow theorem, and one which does not hinge on the infinite extendibility of geodesics. It turns out that, in locally compact spaces, one may substitute for the notion of infinite extendibility, that of extendibility to a closed interval. This version is essentially due to Cohn-Vossen [C-V]; also see [Bu3, p. 4]. (In these references, condition (i) below is not discussed explicitly, but the proof suffices for the theorem as stated here.)

THEOREM 5 [Cohn-Vossen]. *In a locally compact, interior metric space M, the following are equivalent: (i) every halfopen minimizing geodesic from a base point extends to a closed interval; (ii) every halfopen geodesic extends to a closed interval; (iii) bounded closed subsets are compact; (iv) M is complete. Any of these implies: (v) M is a geodesic space (i.e., any two points may be joined by a shortest curve).*

Now the standard proof of the Hadamard-Cartan theorem may be adapted to give:

THEOREM 6. *In a locally compact, complete geodesic space without conjugate points, each homotopy class of curves between two given points contains exactly one geodesic.*

To do this, we modify the covering lemma. Say that a space M has *neighborhoods of radial uniqueness* if every point m is the center of a metric ball B, each of whose points can be joined to m by a unique minimizing geodesic (necessarily in B) and by no other geodesic in B. The proof of the following is entirely standard.

LEMMA 2. *Let M and $\overline{M}$ be complete geodesic spaces. If M has neighborhoods of radial uniqueness, then any local isometry of $\overline{M}$ onto M is a covering map.*

Proof of Theorem 6. Since M has no conjugate points, a sufficiently small metric ball B around m is a neighborhood of radial uniqueness. This fact and local compactness imply that the minimizing geodesic to m in B varies continuously with its endpoint. Thus B is contractible and covering space theory applies to M. Now it suffices to show that the endpoint map $\exp_m$ is a covering map.

By assumption, $\exp_m$ is a local homeomorphism from $(\mathbf{G}_m, \mathbf{d})$ onto M. Now define a new metric $\mathbf{d}^*$ on $\mathbf{G}_m$ by requiring that $\exp_m$ be a local isometry of $(\mathbf{G}_m, \mathbf{d}^*)$ onto M and $(\mathbf{G}_m, \mathbf{d}^*)$ be interior. (This will agree with the metric $\mathbf{d}^*$ of the previous section if M is locally convex, but in general they will be different.) Thus we now take $\mathbf{d}^*(\gamma, \sigma)$ to be the infimum of lengths of curves in M that are the endpoint curves of curves in $(\mathbf{G}_m, \mathbf{d})$ from γ to σ. By Lemma 2, it only remains to show that $(\mathbf{G}_m, \mathbf{d}^*)$ is a complete geodesic space. Note that $(\mathbf{G}_m, \mathbf{d}^*)$ is locally compact, being locally homeomorphic to M. Choose the constant geodesic at m as a basepoint in $\mathbf{G}_m$. A geodesic starting at m in $(\mathbf{G}_m, \mathbf{d}^*)$ projects under $\exp_m$ to a geodesic starting at m in M. Since the latter can be extended to a closed interval by the completeness of M, so can the former. By Theorem 5, since $(\mathbf{G}_m, \mathbf{d}^*)$ satisfies (i), it satisfies (iv) and (v). $\square$

Remark 1. It is easily seen that Cohn-Vossen's theorem does not extend to spaces that are not locally compact. Indeed, an ellipsoid in Hilbert space, the lengths of whose axes are strictly decreasing and bounded above zero, satisfies all but (iii) and (v). The graph of $z = \cos x \cos(1/y)$ for $-\pi/2 \leqslant x \leqslant \pi/2$ and $y > 0$, with the straight line segment from $(-\pi/2, 0, 0)$ to $(\pi/2, 0, 0)$ adjoined, satisfies all but (iii).

Remark 2. A G-space is a locally compact, complete geodesic space which has neighborhoods of bipoint uniqueness, and in which geodesics are infinitely and uniquely extendible. In [Bu2], Busemann studies the Hadamard-Cartan theorem in the setting of G-spaces satisfying a condition that he shows is equivalent to nonpositive curvature in Riemannian manifolds but weaker in G-spaces, and which for brevity we call *local peaklessness* [Bu2, p. 269]. This means that the space is covered by neighborhoods U such that $d(\gamma(t), \sigma)$ is a peakless function for any two shortest curves γ and σ in U. Here, a *peakless* function is one whose values on any interval do not exceed the larger of the two endpoint values, with equality occurring only if the function is constant on the interval. In G-spaces, local peaklessness is equivalent to having *convex capsules* [Bu2, p. 244]. Thus Busemann's theorem is: *A simply connected G-space with convex capsules and domain invariance contains a unique geodesic joining any two of its points.* Our proof of Theorem 2 does not carry over when local convexity is replaced by local peaklessness. However, we have found a different proof that Theorem 2 holds when local convexity is replaced by local compactness and local peaklessness. This fact and Theorem 6 imply, in particular, that the theorem of Busemann just stated holds without the hypothesis of domain invariance.