4. Classification

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.04.2024**

4. Classification

To construct further examples of finite-dimensional representations of Y, we consider tensor products of the evaluation representations $W_m(a)$. In general, if W_1 and W_2 are two representations of Y, the action of Y on the tensor product is given by

$$x.(w_1 \otimes w_2) = \Delta(x)\,(w_1 \otimes w_2)\ ,$$

the action of the right-hand side being that of $Y \otimes Y$ on $W_1 \otimes W_2$. More generally, an r-fold tensor product $W_1 \otimes \cdots \otimes W_r$ is defined using the homomorphism $\Delta^{(r)}: Y \to \underbrace{Y \otimes \cdots \otimes Y}_{r}$ given by

$$\Delta^{(r)} = (\Delta \otimes id \otimes \cdots \otimes id)\Delta^{(r-1)}\ , \qquad \Delta^{(2)} = \Delta\ .$$

Note that, since Δ is co-associative, an equivalent inductive definition is:

$$x.(w_1 \otimes w_2 \otimes \cdots \otimes w_r) = \Delta(x)\,\big(w_1 \otimes (w_2 \otimes \cdots \otimes w_r)\big)\ .$$

Our first main result can now be stated as follows.

THEOREM 4.1. *A tensor product* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is an irreducible representation of* Y *if and only if the strings* $S_{m_i}(a_i)$ *are in general position.*

The proof is in several steps. We begin by analyzing the tensor product $W_m(a) \otimes W_n(b)$ of two evaluation representations. Recall that, as representations of $\mathfrak{sl}_2$, we have

$$W_m(a) \otimes W_n(b) \cong W_{m+n} \oplus W_{m+n-2} \oplus \cdots \oplus W_{|m-n|}\ .$$

We shall refer to the copy of W_{m+n} inside $W_m(a) \otimes W_n(b)$ as its highest component.

The following result proves Theorem 4.1 in the case $r = 2$.

PROPOSITION 4.2. *(a) The tensor product* $W_m(a) \otimes W_n(b)$ *has a proper Y-subrepresentation not containing the highest component if and only if*

$$a - b = \frac{1}{2}(m+n) - p + 1$$

for some $0 < p \leqslant \min\{m, n\}$.

(b) The highest component generates a proper Y-subrepresentation of $W_m(a) \otimes W_n(b)$ *if and only if*

$$b - a = \frac{1}{2}(m+n) - p + 1$$

for some $0 < p \leqslant \min\{m, n\}$.

We need two preliminary lemmas. Let Ω_q denote a highest weight vector (for $\mathfrak{sl}_2$) in the component W_{m+n-2q} of $W_m(a) \otimes W_n(b)$.

LEMMA 4.3. *If* $a - b = \frac{1}{2}(m+n) - p + 1$ *for some* $0 < p \leqslant \min\{m, n\}$, *then*

$$J(h).\Omega_p \in \operatorname{span}\{\Omega_p\} ;$$

$$J(x^+).\Omega_p = 0 ;$$

$$J(x^-).\Omega_p \in \operatorname{span}\{\Omega_{p+1}, x^-.\Omega_p\} .$$

Proof. The vector Ω_p is given by

$$\Omega_p = \sum_{i=0}^{p} (-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!} e_{m-i} \otimes e_{n-p+i} .$$

(To verify this, it is enough to check that Ω_p has the correct weight and that $x^+.\Omega_p = 0$. We omit the simple computation.) From (1.1) we find

$$\Delta(J(x^+)) = J(x^+) \otimes 1 + 1 \otimes J(x^+) - \frac{1}{2} x^+ \otimes h + \frac{1}{2} h \otimes x^+ .$$

Hence,

$$J(x^+).\Omega_p$$

$$= \sum_{i=0}^{p} (-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!} \left(\left(a - \frac{1}{2}(n-2p+2i) \right) (m-i+1) e_{m-i+1} \otimes e_{n-p+i} \right.$$

$$\left. + \left(b + \frac{1}{2}(m-2i) \right) (n-p+i+1) e_{m-i} \otimes e_{n-p+i+1} \right) .$$

The coefficient of $e_{m-i} \otimes e_{n-p+i+1}$ is

$$(-1)^i \frac{(m-i)!(n-p+i)!}{m!(n-p)!} \left(b + \frac{1}{2}(m-2i) \right) (n-p+i+1)$$

$$+(-1)^{i+1}\frac{(m-i-1)!(n-p+i+1)!}{m!(n-p)!}\left(a-\frac{1}{2}(n-2p+2i+2)\right)(m-i)$$

$$=(-1)^{i}\frac{(m-i-1)!(n-p+i)!}{m!(n-p)!}(m-i)(n-p+i+1)\left(b-a+\frac{1}{2}(m+n)-p+1\right),$$

which is zero by our assumption on $a-b$.

The proof of the statements involving $J(h)$ and $J(x^-)$ is similar. We omit the details.

Similar arguments prove the second lemma. Again, we shall omit the details.

LEMMA 4.4. *For any* $0 \leqslant q \leqslant \min\{m, n\}$, *we have*

$$J(h).\Omega_q \in \operatorname{span}\{\Omega_q, x^-.\Omega_{q-1}\};$$
$$J(x^+).\Omega_q \in \operatorname{span}\{\Omega_{q-1}\};$$
$$J(x^-).\Omega_q \in \operatorname{span}\{\Omega_{q+1}, x^-.\Omega_q, (x^-)^2.\Omega_{q-1}\}.$$

Proof of Proposition 4.2.

(a) Suppose that $a-b=\frac{1}{2}(m+n)-p+1$ for some $0<p\leqslant\min\{m,n\}$. We shall prove that

$$V = W_{m+n-2p}\oplus\cdots\oplus W_{|m-n|}$$

is a Y-subrepresentation of $W_m(a)\otimes W_n(b)$. It is enough to show that $(x^-)^r.\Omega_q\in V$ if $p\leqslant q\leqslant\min\{m,n\}$ and $0\leqslant r\leqslant m+n-2q$. We prove this by induction on r. If $r=0$ there is nothing to prove. For any $r\geqslant 1$, we have

$$\begin{aligned}
J(h).(x^-)^r.\Omega_q &= -2J(x^-).(x^-)^{r-1}.\Omega_q + x^-.J(h).(x^-)^{r-1}.\Omega_q\,;\\
J(x^+).(x^-)^r.\Omega_q &= J(h).(x^-)^{r-1}.\Omega_q + x^-.J(x^+).(x^-)^{r-1}.\Omega_q\,;\\
J(x^-).(x^-)^r.\Omega_q &= (x^-)^r.J(x^-).\Omega_q\,.
\end{aligned}$$

The induction hypothesis, together with Lemmas 4.3 and 4.4, shows that the right-hand sides of these formulas are elements of V.

For the converse, suppose that V is a proper subrepresentation of $W_m(a)\otimes W_n(b)$ which does not contain the highest component. Then, for some $0<p\leqslant\min\{m,n\}$, we shall have $\Omega_p\in V$ but $\Omega_q\notin V$ if $q<p$. Then, $J(x^+).\Omega_p=0$, and by the computation in the proof of Lemma 4.3, this implies that $a-b=\frac{1}{2}(m+n)-p+1$.

(b) We shall deduce the second part of the Proposition from the first part using duality. By Corollary 2.9, we have

$$(W_m(a) \otimes W_n(b))^* \cong W_m(-a) \otimes W_n(-b) .$$

Hence, V is a proper subrepresentation of $W_m(a) \otimes W_n(b)$ containing the highest component if and only if the annihilator V^o of V is a proper subrepresentation of $W_m(-a) \otimes W_n(-b)$ *not* containing the highest component. By part (a), $W_m(-a) \otimes W_n(-b)$ has such a subrepresentation if and only if $b - a = \frac{1}{2}(m+n) - p + 1$ for some $0 < p \leqslant \min\{m, n\}$.

Proposition 4.2 can be made more precise.

PROPOSITION 4.5. *Let* $W = W_m(a) \otimes W_n(b), 0 < p \leqslant \min\{m, n\}$. *If* $| a - b | = \frac{1}{2}(m+n) - p + 1$, *then* W *has a unique proper subrepresentation* V. *In fact:*

(a) if $a - b = \frac{1}{2}(m+n) - p + 1$, *we have*

$$V \cong W_{m-p}\left(a + \frac{1}{2}p\right) \otimes W_{n-p}\left(b - \frac{1}{2}p\right) ,$$

$$W/V \cong W_{p-1}\left(a - \frac{1}{2}(m-p+1)\right) \otimes W_{m+n-p+1}\left(b + \frac{1}{2}(m-p+1)\right) ,$$

and, as a representation of $\mathfrak{sl}_2$.

$$V \cong W_{m+n-2p} \oplus \cdots \oplus W_{|m-n|} ;$$

(b) if $b - a = \frac{1}{2}(m+n) - p + 1$, *then*

$$V \cong W_{p-1}\left(a + \frac{1}{2}(m-p+1)\right) \otimes W_{m+n-p+1}\left(b - \frac{1}{2}(m-p+1)\right) ,$$

$$W/V \cong W_{m-p}\left(a - \frac{1}{2}p\right) \otimes W_{n-p}\left(b + \frac{1}{2}p\right) ,$$

and, as a representation of $\mathfrak{sl}_2$,

$$V \cong W_{m+n} \oplus \cdots \oplus W_{m+n-2p+2} .$$

The proof of Proposition 4.2 already gives the uniqueness statements and the isomorphism type under $\mathfrak{sl}_2$. The determination of V as a representation of Y is made using Proposition 1.6 and Theorem 2.4. Since we shall not use this result in the proof of the classification theorem, we omit the details.

Note that Proposition 4.5 (in conjunction with Corollary 4.7 below) enables one to determine the composition series of any tensor product of evaluation representations.

Proposition 4.5 has an interesting string-theoretic interpretation. In (4.5)(a), the subrepresentation corresponds to the "annihilation" of the two strings $S_m(a)$ and $S_n(b)$: the intersection of the strings, together with the two nearest neighbour elements, is discarded, leaving two new strings (in exceptional cases, only one string might remain, or the strings might even annihilate each other completely). Note that the two new strings are always non-interacting. The annihilation interaction is illustrated in the following diagram.

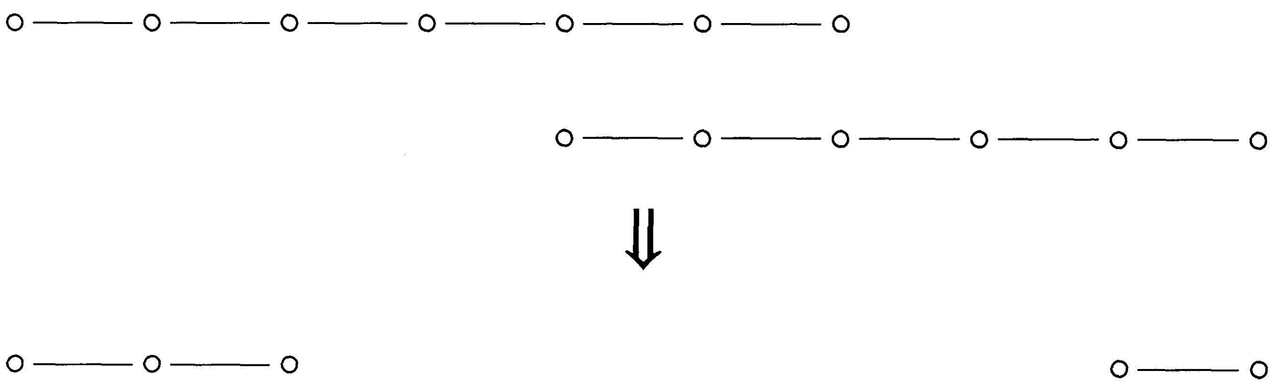

FIGURE 1:
Annihilation of two strings.

The quotient representation in (4.5)(a) corresponds to the "fusion" of the two strings $S_m(a)$ and $S_n(b)$: the two new strings produced by this operation are those which form the unique decomposition of $S_1 \cup S_2$ into the sum of two non-interacting strings (in exceptional cases, only one new string is produced). The fusion interaction is illustrated in the following diagram.

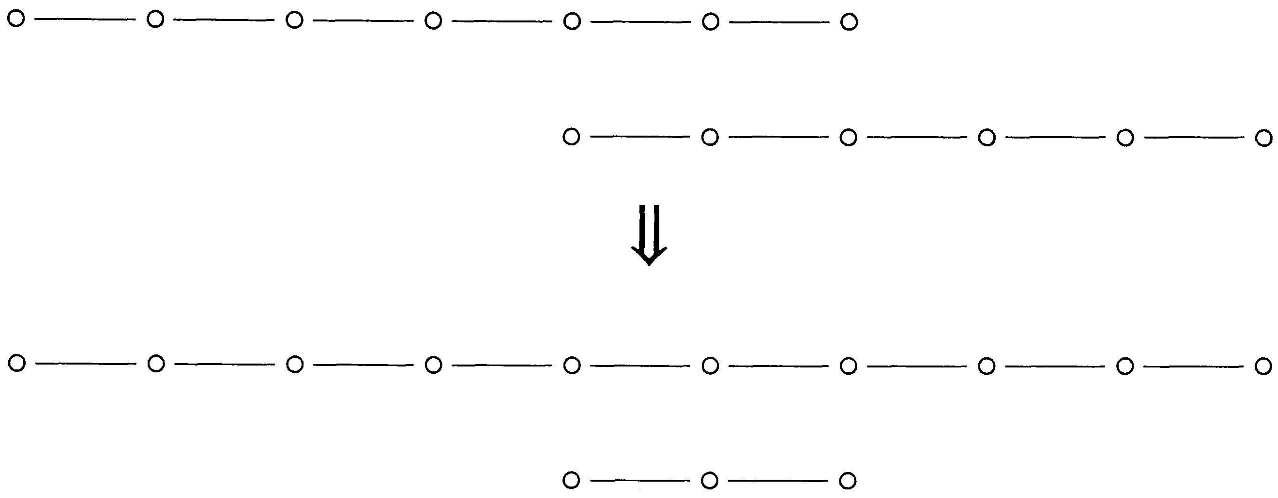

FIGURE 2:
Fusion of two strings.

In (4.5)(b), the roles of the two strings are reversed, and the subrepresentation corresponds to the fusion of the two strings and the quotient to their annihilation.

We now move on to consider tensor products of an arbitrary number of evaluation representations. We begin with:

PROPOSITION 4.6. *If* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is irreducible, then it is highest weight and the polynomial associated to it by Theorem 2.4(b) is the product of the polynomials associated to each factor in the tensor product.*

Proof. It follows from Proposition 1.6(2) that the tensor product of the highest weight vectors in the $W_{m_i}(a_i)$ is a highest weight vector in the tensor product.

As for the second statement, by an easy induction argument using (1.6)(1) and (1.6)(2), we find that

$$\Delta^{(r)}(h_k) = \sum h_{k_1} h_{k_2} \dots h_{k_r} \text{ modulo } \sum\nolimits_{p \geq 0} Y \otimes Yx_p^+ + Yx_p^+ \otimes Y$$

where the first sum is over all r-tuples $k_1, k_2, \dots, k_r$ such that $k_i \geqslant -1$ and $\sum k_i = k - r + 1$ (h_{-1} is interpreted as the identity element 1). Hence, the eigenvalue of h_k on the highest weight vector in $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is, by Proposition 2.6(a),

$$d_k = \sum_{\{k_i\}} \prod_{i=1}^{r} m_i\left(a + \frac{1}{2}m_i - \frac{1}{2}\right)^{k_i}.$$

It is easy to see that this is equal to the coefficient of u^{-k-1} in the product

$$\prod_{i=1}^{r}\left(1 + \sum_{k_i=0}^{\infty} m_i\left(a + \frac{1}{2}m_i - \frac{1}{2}\right)^{k_i} u^{-k_i-1}\right) = \prod_{i=1}^{r} \frac{P_i(u+1)}{P_i(u)},$$

where P_i is the polynomial associated to the representation $W_{m_i}(a_i)$. This completes the proof.

COROLLARY 4.7. *If* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is irreducible, then it is unchanged, up to isomorphism, by any permutation of the factors in the tensor product.*

Proof. Let $V = \bigotimes_{i=1}^{r} W_{m_i}(a_i)$ and let V' be the result of applying some permutation to the factors in the tensor product. Applying the same permutation to the highest weight vector in V gives a highest weight vector in V' of the same weight. It follows from Proposition 2.3 that V is isomorphic to a subquotient of V'. Since V and V' have the same dimension, they must be isomorphic.

Remark. It is *not* true that the permutation of the factors is an isomorphism $V \cong V'$ of representations of Y.

We can now prove the "only if" half of Theorem 4.1. Suppose that some pair of strings $S_{m_j}(a_j)$ and $S_{m_k}(a_k)$ are interacting. Then, by Corollary 4.7, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is isomorphic to a tensor product in which $S_{m_j}(a_j)$ and $S_{m_k}(a_k)$ are adjacent. By Proposition 4.2, the latter representation is reducible.

We now turn to the proof of the "if" part of Theorem 4.1. Note first that there is no loss of generality in assuming that $m_1 \leqslant \cdots \leqslant m_r$. Indeed, since the strings $S_{m_i}(a_i)$ are assumed to be non-interacting, it follows from (4.2) and (4.7) that the tensor product of any pair of the evaluation representations $W_{m_i}(a_i)$ is unchanged, up to isomorphism, by an interchange of the two two factors. Hence, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is unchanged, up to isomorphism, by any permutation of its factors, since the permutation can be effected by a sequence of interchanges of nearest neighbours.

We shall assume that $m_1 \leqslant \cdots \leqslant m_r$ for the rest of the proof of (4.1). The main step in the proof is the following result.

PROPOSITION 4.8. *Suppose that the strings* $S_{m_i}(a_i)$, $1 \leqslant i \leqslant r$, *are non-interacting, and that* $m_1 \leqslant \cdots \leqslant m_r$. *Then* $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ *is generated by the tensor product of the highest weight vectors in the* $W_{m_i}(a_i)$.

Assuming this result for a moment, the proof of Theorem 4.1 is completed as follows. Suppose that the strings $S_{m_i}(a_i)$ are non-interacting. Note that, as a representation of $\mathfrak{sl}_2$, $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ contains a unique highest component W_M, $M = \sum m_i$. By (4.2), (4.7) and (4.8), if $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ has a proper subrepresentation V, then V does not contain W_M. But then the annihilator V^o of V is a proper subrepresentation of the dual

$$(\bigotimes_{i=0}^{r} W_{m_i}(a_i))^* \cong \bigotimes_{i=0}^{r} W_{m_i}(-a_i)$$

which does contain its highest component. By (4.2), (4.7) and (4.8) again, this is impossible.

Remark. The following is an interesting alternative argument. By Proposition 2.10, each factor $W_{m_i}(a_i)$ has an invariant bilinear form. If W_1 and W_2 are two representations of Y which have invariant forms $< , >_1$ and $< , >_2$, then there is an invariant bilinear map

$$(W_1 \otimes W_2) \times (W_2 \otimes W_1) \to \mathbf{C}$$

given by

$$< w_1 \otimes w_2, w_2' \otimes w_1' > \; = \; < w_1, w_1' >_1 < w_2, w_2' >_2 .$$

(The change of order is necessary because Y is not co-commutative.) In particular, if $W_1 \otimes W_2 \cong W_2 \otimes W_1$, then $W_1 \otimes W_2$ has an invariant bilinear form. Using this observation, the fact that $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is unchanged, up to isomorphism, by any permutation of the factors (which follows from (4.2) and (4.7)), and an easy induction, one sees that $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ has an invariant bilinear form. But now a standard argument in the theory of Lie algebras shows that a highest weight representation which carries a non-zero invariant bilinear form is irreducible.

We must now give the

Proof of Proposition 4.8. By induction on r. The result is known if $r = 1$ or 2.

We first prove that $W = \bigotimes_{i=1}^{r} W_{n_i}(a_i)$ is generated by the vector $e_0 \otimes \Omega'$, where $\Omega' = e_{m_2} \otimes \cdots \otimes e_{m_r}$ is the highest weight vector in $W' = \bigotimes_{i=2}^{r} W_{m_i}(a_i)$. By the induction hypothesis, W' is generated by Ω'. From Proposition 1.11, for any $w' \in W'$, there exists $y \in N^-$ such that $y.\Omega' = w'$. Then

$$y.(e_0 \otimes \Omega') = \Delta(y)\,(e_0 \otimes \Omega') = e_0 \otimes w' ,$$

where the last equality follows from Proposition 1.6(3) and the fact that $x_k^- . e_0 = 0$. Hence, $e_0 \otimes W' \subseteq Y.(e_0 \otimes \Omega')$. Now an easy induction on i proves that $e_i \otimes W' \subseteq Y.(e_0 \otimes \Omega')$ for $0 \leqslant i \leqslant m_1$: for the inductive step one uses the fact that

$$e_{i+1} \otimes W' = x^+ . e_i \otimes W' \subseteq e_i \otimes W' + x^+ .(e_i \otimes W') .$$

This proves our assertion.

We now prove that $e_0 \otimes \Omega' \in Y.\Omega$, where $\Omega = e_{m_1} \otimes \cdots \otimes e_{m_r}$. For any $i > 0$, consider the equations

$$(4.9) \qquad x_k^- .(e_i \otimes \Omega') = \Big(\sum_{p=0}^{k} b_1^p d_{k-p-1,1} x^- . e_i\Big) \otimes \Omega' + e_i \otimes x_k^- . \Omega' ,$$

for $k = 0, \ldots, r-1$, where $b_1 = a_1 - \frac{1}{2} m_1 + i - \frac{1}{2}$, $d_{k,1}$ is the eigenvalue of h_k on Ω' (and $d_{-1,1} = 0$). Equation (4.9) follows from Proposition 1.6 (3), using the fact that Ω' is a highest weight vector for Y. More generally, iterating (4.9), we find that

$$(4.10) \qquad x_k^- .(e_i \otimes \Omega') = \sum_{j=1}^{r} A_{k,j} e_i \otimes \cdots \otimes x^- . e_{m_j} \otimes \cdots \otimes e_{m_r} ,$$

where

$$A_{k,j} = \sum_{p=0}^{k} b_j^p d_{k-p-1,j} ,$$

$$b_j = a_j + \frac{1}{2} m_j - \frac{1}{2} \quad \text{for} \quad j \geqslant 2 ,$$

and $d_{k,j}$ is the eigenvalue of h_k on $e_{m_{j+1}} \otimes \cdots \otimes e_{m_r}$ (and $d_{-1,j} = 1$).

Using Proposition 1.6(1), one sees that

$$d_{k,j} = m_{j+1} A_{k,j+1} + d_{k,j+1}$$

so we are in the situation of (3.6). Assuming Proposition 3.7, which has yet to be proved, we have

$$\det A = \prod_{1 \leqslant k < j \leqslant r} (b_j - b_k - m_j) .$$

Since the strings $S_{m_j}(a_j)$ are non-interacting, this determinant is non-zero. For, $b_j = b_k + m_j$ for some $j > k > 1$ implies that $a_j - a_k = \frac{1}{2}(m_j + m_k)$, which is impossible; and $b_j = b_1 + m_j$ implies that $a_j - a_1 = \frac{1}{2}(m_j - m_1) - i = \frac{1}{2}|m_j - m_1| - i$, which is also impossible since $i > 0$.

Hence, equation (4.10) implies that e_{i-1} is a linear combination of the elements $x_k^- .(e_i \otimes \Omega')$ for $0 \leqslant k \leqslant r-1$. An obvious (downward) induction now proves that $e_i \otimes \Omega' \in Y.(e_{m_1} \otimes \Omega') = Y.\Omega$ for all $i \geqslant 0$. In particular, we have proved that $e_0 \otimes \Omega' \in Y.\Omega$.

All that remains is to prove Proposition 3.7. We show first that $b_j - b_{j-1} - m_j$ is a root of $\det A$ for $2 \leqslant j \leqslant r$. In fact, we shall prove that, if $b_j - b_{j-1} - m_j = 0$, then the j-th and $(j+1)$-th columns of the matrix A are the same. To begin with, $A_{0,j} = A_{0,j-1} = 1$. Proceeding by induction on k and using (3.6), we have

$$\begin{aligned}
A_{k+1,j-1} &= b_{j-1}A_{k,j-1} + d_{k,j-1} \\
&= b_{j-1}A_{k,j} + d_{k,j-1} \\
&= (b_j - m_j)A_{k,j} + d_{k,j-1} \\
&= (b_j - m_j)A_{k,j} + m_jA_{k,j} + d_{k,j} \\
&= b_jA_{k,j} + d_{k,j} \\
&= A_{k+1,j} ,
\end{aligned}$$

which proves our assertion.

If $j > k$ is any pair of indices, there is a permutation σ of $\{1, ..., r\}$ such that $\sigma(1) = 1$ and $\sigma(k) = \sigma(j) - 1$. Let Ω'_σ be the result of applying σ to the factors in Ω', and define W_σ and W'_σ similarly. As we remarked earlier, W' and W'_σ are isomorphic as representations of Y, and the isomorphism must preserve highest weight vectors. Hence, there is an isomorphism $W \cong W_\sigma$ which takes $e_i \otimes \Omega'$ to $e_i \otimes \Omega'_\sigma$ for all i. Hence,

$$\{x_0^- .(e_i \otimes \Omega'), ..., x_{r-1}^- .(e_i \otimes \Omega')\}$$

is linearly dependent if and only if

$$\{x_0^- .(e_i \otimes \Omega'_\sigma), ..., x_{r-1}^- .(e_i \otimes \Omega'_\sigma)\}$$

is linearly dependent. By (4.10), the first condition holds if and only if $\det A = 0$, and the second if and only if $\det A_\sigma = 0$, where A_σ is the matrix obtained by applying σ to the parameters $a_1, \ldots, a_r, m_1, \ldots, m_r$. This implies that $b_j - b_k - m_j$ is a root of $\det A$ if and only if $b_{\sigma(j)} - b_{\sigma(k)} - m_{\sigma(j)}$ is a root of $\det A_\sigma$, and this is true by the first part of the argument.

We have now proved that $b_j - b_k - m_j$ is a root of $\det A$ for all $j > k$. This proves Proposition 3.7 in the case of interest to us, namely when the m_j are positive integers. But since (3.7) is a polynomial identity, it holds in general.

The proof of Theorem 4.1 is now complete.

The following result completes the classification of the finite-dimensional irreducible representations of Y.

THEOREM 4.11. *(a) Every finite-dimensional irreducible representation of Y is isomorphic to a tensor product of evaluation representations $W_m(a)$.*

(b) Two irreducible tensor products of evaluation representations are isomorphic as representations of Y if and only if one is obtained from the other by a permutation of the factors in the tensor product.

Proof. (a) Let V be a finite-dimensional irreducible representation of Y. Let P be the polynomial corresponding to V in Theorem 2.4. The roots of P form a set with multiplicities which, by (3.5), can be written as a union of non-interacting strings. Let $S_{m_i}(a_i)$ be the strings which occur (the m_i, a_i are not necessarily distinct). By (4.1) and (4.6), the tensor product $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$ is irreducible and has P as its associated polynomial (by (4.7), the order of the factors in the tensor product is immaterial). By Theorem 2.4, V is isomorphic to $\bigotimes_{i=1}^{r} W_{m_i}(a_i)$.

(b) Suppose that

$$\otimes W_{m_i}(a_i) \cong \otimes W_{n_j}(b_j) .$$

are irreducible representations of Y. Then, both tensor products are associated to the same polynomial P. The $S_{m_i}(a_i)$ and the $S_{n_j}(b_j)$ both give decompositions of the roots of P into sets of non-interacting strings. By Proposition 3.5, the decompositions are the same. This means that the factors $W_{m_i}(a_i)$ and $W_{n_j}(b_j)$ are the same up to a permutation.