3. Solvable groups

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Observation 2.2. There is a non-locally linear action of a finite group on a homology four sphere with exactly one fixed point.

Proof. Take the one fixed point action of A_5 on the Poincaré's sphere constructed in [11], remove the fixed point and multiply the remaining homology disk by the unit interval to obtain a four homology disk on which the product action has no fixed points. One can extend this action to the one point compactification to obtain a homology S^4 on which A_5 acts fixing only the point at infinity.

The main tool in the proof of Theorem 2.1. will be the classical result due to Smith (see [19]);

THEOREM 2.3. *Let Z/p, p a prime, act on a Z/p homology S^n, then the fixed point set is a Z/p homology S^k; if p is odd, $n - k$ is even.*

3. SOLVABLE GROUPS

In the four dimensional case it is easy to deduce from Theorem 2.3. the Corollary:

COROLLARY 3.1. *Let G be a solvable group acting locally linearly and orientation preserving on Σ, then the fixed point set is a sphere.*

Proof of the Corollary. Let $\{ I \} = H_0 \subset H_1 \subset H_2 \subset G$ be a composition series such that every H_{i+1} is normal in H_i and the quotients are cyclic of prime order p_i. By Smith theorem $X = \mathrm{Fix}(H_i)$ is a Z/p homology sphere, the action is not trivial so X cannot be the whole Σ; nor can it be 3-dimensional, for otherwise some element of H_1 would interchange the two components of $\Sigma - X$ and so reverse the orientation. Hence X has to be of dimension less than or equal to 2 and so a topological sphere.

For $i > 1$, $\mathrm{Fix}(H_{i-1})$ is invariant under H_i and the latter's action factorizes through H_i/H_{i-1}, so $\mathrm{Fix}(H_i) = \mathrm{Fix}\big(H_{i-1}/H_i \mid \mathrm{Fix}(H_{i-1})\big)$; applying repeatedly the argument above and using the fact that now all the spaces involved are spheres, the statement follows.

If $x_0 \in \Sigma^G$, the fixed set of G on Σ, the assumption of local linearity gives a representation $G \xrightarrow{\rho} SO(4)$, faithful since G acts effectively, this allows us to think of G as a finite subgroup of $SO(4)$ and to study it we look at the central extension:

3.2 $$0 \to C_2 \to SO(4) \to SO(3) \times SO(3) \to 0$$

where $\pi = (\pi_+, \pi_-)$ is given by the representation onto the self-dual and anti-self-dual forms in R^4, and C_2 is $\{\pm I\}$ the center of $SO(4)$.

Observe that $\pi^{-1}(\Delta)$, where Δ is the diagonal in $SO(3) \times SO(3)$, is the image of the "suspension" map from $O(3)$ into $SO(4)$:

$$M \to \begin{pmatrix} \det M & 0 \\ 0 & M \end{pmatrix}$$

We state now two elementary facts which will become useful in the following;

LEMMA 3.3. *If* $\alpha \in SO(4)$ *has at least one eigenvalue* $= 1$ *then its image* $\pi(\alpha) = (\alpha_+, \alpha_-)$ *in* $SO(3) \times SO(3)$ *is conjugate to an element of* Δ, *i.e.,* $v^{-1}\alpha_+ v = \alpha_-$ *for some* $v \in SO(3)$.

LEMMA 3.4. *The fixed space of an element of* $SO(4)$ *always has even dimension.*

Consider the diagram

3.5
$$\begin{array}{ccccccc}
SO(4) & \xrightarrow{\pi} & & & SO(3) & \times & SO(3) \\
\cup & & & & \cup & & \cup \\
G \cdot C_2 = \tilde{G} & \xrightarrow{\pi} & G_0 & \subset & G_1 & \times & G_2 \\
j \cup & & & & & & \\
G & & & & & &
\end{array}$$

where the G_is $(i = 1, 2)$ are the images of the projections π_i of G_0 into the two $SO(3)$s; j is either the identity or the inclusion of a subgroup of index 2 in $\tilde{G} = \pi^{-1}(\pi(G))$ in the latter case $\pi \circ j$ appear as G_i. Luckily, finite subgroups of $SO(3)$ are well known (see e.g. [20]): they can be divided into four types:

i. cyclic groups C_n,

ii. dihedral groups D_{2m},

iii. the tetrahedral group,

iv. the octahedral group,

v. the icosahedral group.

All the first four types consist of solvable groups. It is easy to show that the class of solvable groups is closed under the operations of taking products, subgroups and central extensions, so G falls in the hypothesis of Corollary 3.1. in all cases, except the one in which at least one G_i is the icosahedral group. This is isomorphic to A_5, the alternating group on five letters and this identification will be fixed from now on.

4. Non solvable groups

We will prove Theorem 2.1 case by case. We start with the Lemma:

LEMMA 4.1. *If G contains C_2, then* $\text{Fix}(G)$ *is* S^0.

Proof. $\text{Fix}(G) = \text{Fix}(G/C_2\text{Fix}(C_2))$. $\text{Fix}(C_2)$ is a homology sphere by Smith's theorem and is zero dimensional since around the chosen fixed point the non trivial element of C_2 acts like the matrix $-I$, which has an isolated fixed point. The action of G/C_2 on S^0 has to be trivial since the fixed point set is required not to be empty.

By renumbering the factors and changing basis if necessary, we may assume G_2 equal to A_5, with $G_2 \xrightarrow{i} SO(3)$ the standard representation of A_5. Then G_0 is a subgroup of $G_1 \times A_5$ mapping onto both factors and to study it in more detail we look at the kernel of the second projection: $G_0 \xrightarrow{\pi_2} A_5$. This subgroup consists of elements of the form (k, I) with $k \in G_1$; we denote it by K_1.

For convenience we distinguish three cases:

Case 1. K_1 is a non-trivial subgroup of $SO(3)$, not isomorphic to A_5,

Case 2. K_1 is isomorphic to A_5,

Case 3. K_1 is trivial.

Proof in case 1. The surjection $G \to A_5$ has non trivial kernel $K = j^{-1}(\pi^{-1}(K_1)) \subset G$, this group is solvable since K_1 is, π is a central extension and j is an injection. By Corollary 3.1., $\text{Fix}(K)$ is a sphere of dimension 2 and $\text{Fix}(G)$ is the fixed point set of an A_5 acting on it, so it is easy to see that the only actions admitting some fixed points are the trivial ones.

Proof in case 2. Since A_5 is not properly contained in any finite subgroup of $SO(3)$, K_1 has to be equal to the whole G_1.

So $G_0 \subset A_5 \times A_5 \subset SO(3) \times SO(3)$ and contains $K_1 = A_5 \times \{I\}$, it follows that G_0 is the whole $A_5 \times A_5$. Observe that the two inclusions of A_5 in $SO(3)$ do not necessarily agree.