

# Section 1. Proof of Theorem 2

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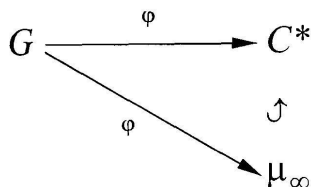
## **Haftungsausschluss**

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$$c(\rho \circ f) = f^*(c \cdot (\rho)).$$

CH2.  $c \cdot (\rho_1 \oplus \rho_2) = c \cdot (\rho_1) \cdot c \cdot (\rho_2).$

CH3.  $c_1: \text{Hom}(G, \mathbf{C}^*) \rightarrow H^2(G, \mathbf{Z})$  is an isomorphism and can be described as follows: For  $\varphi \in \text{Hom}(G, \mathbf{C}^*)$ , let  $\varphi$  also denote its unique factorization



Now  $c_1(\varphi) = \varphi^*(u).$

Remark. As shown in [7], CH1, CH2 and CH3 uniquely determine the Chern classes defined by  $u$ . As different choices of  $u$  clearly defines different Chern classes (just observe that

$$H^2(\mu_\infty, \mathbf{Z}) \cong \varinjlim H^2(G_i, \mathbf{Z}),$$

the limit taken over all finite cyclic subgroups), there is a one-to-one correspondence between Chern classes on finite groups and  $\hat{\mathbf{Z}}$  generators of  $H^2(\mu_\infty, \mathbf{Z})$ .

This paper has been organized as follows.

Theorem 2 is proved in Section 1, Theorem 4 in Section 2, and Theorem 5 in Section 3. Proposition 3 i) was proved in [7], and the remaining part of this proposition can be obtained similarly.

Finally, in Section 4 it is shown that there exists a very simple extension of the theory of Chern classes on finite groups to locally finite groups.

I would like to thank Jørgen Tornehave for a helpful conversation.

### SECTION 1. PROOF OF THEOREM 2

CH1 is quite trivial, so let me first prove CH2. Let  $\dim \rho_i = n_i$ ,  $\dim \rho = n$ , so that  $n_1 + n_2 = n$ . By assumption,  $\rho$  factors through the parabolic subgroup  $P = P(k_p)$

$$P = \begin{pmatrix} & n_1 & & n_2 \\ * & & * & \\ & & & \\ 0 & & & * \end{pmatrix}$$

which is isomorphic to a semi-direct product of  $Gl_{n_1}(\bar{k}_p) \times Gl_{n_2}(\bar{k}_p)$  acting on a unipotent subgroup  $U$ .

As  $U$  is a direct limit of  $p$ -groups,

$$H^k(U, \hat{\mathbf{Z}}_l) = 0 \quad \text{for } k > 0.$$

Thus

$$\begin{aligned} H^*(P, \hat{\mathbf{Z}}_l) &\cong H^*(Gl_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \otimes H^*(Gl_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \\ &\cong P(\alpha_1, \dots, \alpha_{n_1}) \otimes P(\beta_1, \dots, \beta_{n_2}). \end{aligned}$$

Let

$$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, \dots, \sigma_n)$$

and

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_n).$$

As  $T_n(\bar{k}_p) \cong T_{n_1}(\bar{k}_p) \times T_{n_2}(\bar{k}_p)$ , I shall consider

$$H^*(T_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_{n_1})$$

and

$$H^*(T_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_{n_1+1}, \dots, x_n)$$

as contained in  $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ . Furthermore, as all restriction maps are injective, I shall view  $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$  and  $H^*(Gl_{n_i}(\bar{k}_p), \hat{\mathbf{Z}}_l)$ ,  $i = 1, 2$ , as subspaces of  $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ . Thus

$\alpha_i$  = the  $i$ 'th elementary symmetric polynomial in  $x_1, \dots, x_{n_1}$

$\beta_i$  = the  $i$ 'th elementary symmetric polynomial in  $x_{n_1+1}, \dots, x_n$

$\sigma_i$  = the  $i$ 'th elementary symmetric polynomial in  $x_1, \dots, x_n$ .

Furthermore, the formula

$$c \cdot (\rho_1 \oplus \rho_2) = c \cdot (\rho_1 \oplus \rho_2)$$

is equivalent to

$$1 + \sigma_1 t + \dots + \sigma_n t^n = (1 + \alpha_1 t + \dots + \alpha_{n_1} t^{n_1}) \cdot (1 + \beta_1 t + \dots + \beta_{n_2} t^{n_2}),$$

and this follows from the identity

$$\begin{aligned} \sum_{i=0}^n \sigma_i t^i &= \prod_{i=1}^n (1 + tx_i) = \prod_{i=1}^{n_1} (1 + tx_i) \cdot \prod_{i=n_1+1}^{n_2} (1 + tx_i) \\ &= \left( \sum_{i=0}^{n_1} \alpha_i t^i \right) \left( \sum_{i=0}^{n_2} \beta_i \cdot t^i \right). \end{aligned}$$

To prove CH3, observe that for  $G$  locally finite the homology groups  $H_i(G, \mathbf{Z})$  are all torsion groups for  $i > 0$  as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups  $G_k$  of  $G$  such that  $\varinjlim G_k = G$ . Now, by the universal coefficient theorem,

$$0 \rightarrow \text{Ext}_{\mathbf{Z}}^1(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow H^2(G, \hat{\mathbf{Q}}_l) \rightarrow \text{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow 0$$

is exact ( $\hat{\mathbf{Q}}_l$  is the quotient field of  $\hat{\mathbf{Z}}_l$ ) so it follows that  $H^2(G, \hat{\mathbf{Q}}_l) = 0$  as  $\hat{\mathbf{Q}}_l$  is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as  $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$ , where  $C_{l^\infty}$  is the injective hull of a cyclic  $l$ -group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as  $G$  is locally finite and  $\bigoplus_{l \neq p} C_{l^\infty}$  is the torsion subgroup of  $\prod_{l \neq p} C_{l^\infty}$ .

SECTION 2. PROOF OF THEOREM 4

Let  $G$  be a given finite group of order  $|G|$  and

$$\rho: G \rightarrow Gl_n(\mathbf{C})$$

a complex representation.

Choose  $q$  to be a power of a prime number  $p$  different from  $l$  such that

$$q \equiv 1 \pmod{|G|}$$

Define

$$\phi: Gl_n(q) \rightarrow \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^n e_p(\lambda_i)$$