

K NON-DYADIC

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

K NON-DYADIC

We begin with a well known lemma, valid for an arbitrary field of characteristic 0.

LEMMA. Let K_0 be a subfield of K which contains $\mathbf{Q}^c \cap K$. Then

$$S(K) = K \otimes S(K_0)$$

where $K \otimes S(K_0)$ denotes the subgroup of $B(K)$ obtained from $S(K_0)$ by extension of scalars.

Proof. See Prop. 4.6 in [Y]; a proof of a more general result can be found in [M]. Since the proof is short, we give it here. We can assume $K_0 = \mathbf{Q}^c \cap K$. Let $\beta \in S(K)$ and let A be a Schur algebra in β , i.e. a simple component of some group algebra KG with Brauer class $[A] = \beta$. Then A is also a direct summand of $K \otimes A_0$ for some simple component A_0 of K_0G . But the center of A_0 is a sub-cyclotomic extension of K_0 (see exercise 9.15, [I], e.g.), so is K_0 since it is also contained in K . Thus $[A_0] \in S(K_0)$ and $A = K \otimes A_0$. It follows that $S(K) \subseteq K \otimes S(K_0)$ and the reverse inclusion is obvious. \square

This lemma allows us to assume, from now on, that K is a sub-cyclotomic extension of \mathbf{Q}_p , i.e. a (finite) abelian extension of \mathbf{Q}_p .

We shall denote the group of roots of unity of a field L by $\mu(L)$. The subgroup of roots of unity of order a power of p , resp. of order relatively prime to p , is denoted by $\mu(L)_p$ resp. $\mu(L)_{p'}$. The group of all roots of unity, i.e. $\mu(\mathbf{Q}_p^c)$, will be denoted by μ , with μ_2 and μ_2' having the obvious meanings.

Assume now that p is odd. Since $\mu(\mathbf{Q}_p)$ is $\cong \mathbf{Z}/p-1$, the root of unity theorem of Benard and Schacher (Th. 6.1, [Y]) and the fact that $B(\mathbf{Q}_p) \cong \mathbf{Q}/\mathbf{Z}$ (see [S], e.g.) imply that $S(\mathbf{Q}_p) \hookrightarrow \mathbf{Z}/p-1$. (In fact this map is an isomorphism). By the theory of central simple algebras over a local field, we can identify $B(K)$ with $H^2(\mathcal{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*)$, (which we also denote by $H^2(\mathbf{Q}_p^c/K)$). By the Brauer-Witt theorem (Cor. 3.11, [Y]), $S(K)$ is thereby identified with the image of the canonical map

$$H^2(\mathcal{G}(\mathbf{Q}_p^c/K), \mu) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_p^c/K), (\mathbf{Q}_p^c)^*), \quad (\mu = \mu(\mathbf{Q}_p^c)),$$

which we denote by $H_c^2(\mathbf{Q}_p^c/K)$. The (cohomological) corestriction map $B(K) \rightarrow B(\mathbf{Q}_p)$ carries $S(K)$ into $S(\mathbf{Q}_p)$ since, on the cocycle level, it takes a cocycle f to a cocycle whose values are products of the values of f (see [W], e.g.). Furthermore the corestriction is injective in this case

(p. 175, [S]) and so $S(K)$ is finite — in fact it is a subgroup of $\mathbb{Z}/p-1$. We may therefore choose a primitive m^{th} root of unity ε_m so that

$$S(K) = H_c^2(\mathbb{Q}_p(\varepsilon_m)/K).$$

We may also assume that $p \mid m$, i.e. that $\varepsilon_p \in \mathbb{Q}_p(\varepsilon_m)$.

We now show that $\mathbb{Q}_p(\varepsilon_m)$ can be replaced by a field L so that L/K is cyclic and totally tamely ramified. First of all, by Lemma 4.1, [Y], we can assume that $\mathbb{Q}_p(\varepsilon_m)$ is the (disjoint) compositum UV of an unramified extension U/K and a totally ramified extension V/K . Since the order of $S(K)$ is relatively prime to p , $S(K)$ is the image $H_c^2(\mathbb{Q}_p(\varepsilon_m)/K)'$ of the canonical map

$$H^2(\mathcal{G}(\mathbb{Q}_p(\varepsilon_m)/K), \mu(\mathbb{Q}_p(\varepsilon_m))_{p'}) \rightarrow H^2(\mathbb{Q}_p(\varepsilon_m)/K).$$

Since UV/V is unramified, $N_{UV/V}(\mu(UV)_{p'}) = \mu(V)_{p'}$, and it follows from the inflation-restriction sequence (see Lemme 1, [F]) that the inflation

$$H^2(\mathcal{G}(V/K), \mu(V)_{p'}) \rightarrow H^2(\mathcal{G}(UV/K), \mu(UV)_{p'})$$

is an isomorphism (since UV/V cyclic implies that $H^2(\mathcal{G}(UV/V), \mu(UV)_{p'}) \cong H^0(\mathcal{G}(UV/V), \mu(UV)_{p'}) = 1$). Thus $S(K) = H_c^2(V/K)'$. Let L/K be the tamely ramified part of V/K . Since the p' roots of unity in a local field are the same as the non-zero elements in the residue class field, $\mu(V)_{p'} = \mu(L)_{p'} = \mu(K)_{p'}$ and so $N_{V/L}(\mu(V)_{p'}) = \mu(L)_{p'}$ because $(V:L)$ is a power of p . Once again the inflation-restriction sequence shows that $S(K) = H_c^2(L/K)'$.

Consider now the cup product pairing

$$(1) \quad \smile : \mathcal{G}(L/K) \times H^2(L/K) \rightarrow K^*/N_{L/K}L^*.$$

See for example pp. 139-140, [C-F]. It is known that there is a "canonical class" $u_{L/K}$ in $H^2(L/K)$ with the property that the map $\sigma \mapsto \sigma \smile u_{L/K}$ is an isomorphism $\mathcal{G}(L/K) \rightarrow K^*/N_{L/K}L^*$. It follows that if σ is a generator of $\mathcal{G}(L/K)$, the map

$$\sigma \smile : H^2(L/K) \rightarrow K^*/N_{L/K}L^*$$

is also an isomorphism. We wish to identify the image of $H_c^2(L/K)'$ under this map. The cohomology class $[f]$ of the cocycle f has image $\prod_{\tau} f(\tau, \sigma) \bmod N_{L/K}L^*$ (Lemme 4, p. 186, [S]). Since $\mathcal{G}(L/K)$ is cyclic with generator σ , every cohomology class with coefficients in an arbitrary $\mathcal{G}(L/K)$ -module A is represented by a cocycle f of the form

$$f(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if } i+j < d, \\ a & \text{if } i+j \geq d. \end{cases}$$

Here $d = |\mathcal{G}(L/K)|$, $0 \leq i, j < d$, and a is an arbitrary element of $A^{\mathcal{G}(L/K)}$. If $A = L^*$, it follows that a class in $H^2(L/K)$ is in $H_c^2(L/K)'$ iff it contains such an f with $a \in \mu(K)_{p'}$. Since it is clear that $\sigma \cup [f] = a \bmod N_{L/K}L^*$, we see that the image of $H_c^2(L/K)'$ is

$$\mu(K)_{p'} N_{L/K}L^* / N_{L/K}L^* \cong \mu(K)_{p'} / \mu(K)_{p'} \cap N_{L/K}L^*.$$

But it is easy to see that $\mu(K)_{p'} \cap N_{L/K}L^* = N_{L/K}\mu(L)_{p'}$ so we have an isomorphism

$$S(K) = \mu(K)_{p'} / N_{L/K}\mu(L)_{p'}$$

depending only on the choice of σ .

We now show that the norm residue symbol

$$v_K = (\ , \mathbf{Q}_p^c/K): K^* \rightarrow \mathcal{G}(\mathbf{Q}_p^c/K)$$

induces an isomorphism of $\mu(K)_{p'} / N_{L/K}\mu(L)_{p'}$ onto $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$. It is clear that the image of $\mu(K)_{p'}$ is contained in $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$. Let $v = (\ , \mathbf{Q}_p^c/\mathbf{Q}_p)$. The diagram

$$\begin{array}{ccc} K^* & \xrightarrow{v_K} & \mathcal{G}(\mathbf{Q}_p^c/K) \\ N_{K/\mathbf{Q}_p} \downarrow & & \downarrow \text{incl.} \\ \mathbf{Q}_p^* & \xrightarrow{v} & \mathcal{G}(\mathbf{Q}_p^c/\mathbf{Q}_p) \end{array}$$

is commutative (Prop. 10, ch. XIII, [S]). Recall now that $\varepsilon_p \in \mathbf{Q}_p(\varepsilon_m)$. It follows that the tame ramification index of L/\mathbf{Q}_p is $p-1$. Therefore if L'/\mathbf{Q}_p is the maximal unramified subextension of L/\mathbf{Q}_p , then $(L:L')$ is a p -power multiple of $p-1$. Since $\mu(L)_{p'} = \mu(L')_{p'}$ and $N_{L'/\mathbf{Q}_p}\mu(L')_{p'} = \mu(\mathbf{Q}_p)_{p'} \cong \mathbf{Z}/p-1$, the kernel κ of the restriction of N_{K/\mathbf{Q}_p} to $\mu(K)_{p'}$ is $\subseteq N_{L/K}\mu(L)_{p'}$. On the other hand if one factors N_{K/\mathbf{Q}_p} through the tame and unramified closures of \mathbf{Q}_p in K , one sees that $N_{L/K}$ on $\mu(K)_{p'}$ is $\varepsilon \mapsto \varepsilon^{e(p^f-1)/(p-1)}$ where e and f are resp. the ramification and inertial indices of K/\mathbf{Q}_p . It follows that $\kappa = N_{L/K}\mu(L)_{p'}$ which is equal to $\ker v_K|_{\mu(K)_{p'}}$ since v is injective.

Now v maps the torsion subgroup $\mu(\mathbf{Q}_p) \cong \mathbf{Z}/p-1$ of \mathbf{Q}_p^* onto the torsion subgroup of $\mathcal{G}(\mathbf{Q}_p^c/\mathbf{Q}_p)$. Furthermore an element $a \in \mathbf{Q}_p^*$ is mapped into $\mathcal{G}(\mathbf{Q}_p^c/K)$ iff $a \in N_{K/\mathbf{Q}_p}K^*$. It follows at once that v maps $N_{K/\mathbf{Q}_p}\mu(K)_{p'}$ isomorphically onto $\text{tor } \mathcal{G}(\mathbf{Q}_p^c/K)$. This proves our main theorem in the case p odd.

K DYADIC

It would be very nice to have a unified proof for the dyadic and non-dyadic cases along the lines of the one above for the non-dyadic case. However that would require a "deflation" $H_c^2(V/K) \cong H_c^2(L/K)$ to some *cyclic* extension L/K in order that the cup product pairing (1) be non-degenerate on both sides. U. Jannsen has shown that this is impossible in general. Since $H^2(L_1/K) = H^2(L_2/K)$ (when inflated to a common extension) if $(L_1:K) = (L_2:K)$, one can try to replace the cyclotomic extension V/K by a some cyclic but possibly non-cyclotomic extension to achieve non-degeneracy. This is done, however, at the expense of losing the identification of $S(K)$ as the subgroup of cyclotomic cocycles. This is essentially what is done in the second half of the following proof.

Since $\mu(\mathbf{Q}_2) = \pm 1$, it follows (as in the non-dyadic case) that $S(K)$ is 1 or ± 1 . Thus to prove the theorem it suffices to show that

$$(2) \quad S(K) \neq 1 \Leftrightarrow -1 \in \mathcal{G}(\mathbf{Q}_2^c/K).$$

Before beginning we recall a few facts about Galois groups of \mathbf{Q}_2^c . Let ε_m be a primitive m^{th} root of unity and write $m = 2^n m'$ where m' is odd. Let f be the smallest integer such that $m' \mid 2^f - 1$. Then if $n \geq 2$,

$$\mathcal{G}(\mathbf{Q}_2(\varepsilon_m)/\mathbf{Q}_2) \cong \mathbf{Z}/f \times (\mathbf{Z}/2^n)^* \cong \mathbf{Z}/f \times \mathbf{Z}/2^{n-2} \times \mathbf{Z}/2.$$

Taking \varprojlim over m one gets

$$(3) \quad \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2.$$

The topological generator 1 of $\hat{\mathbf{Z}}$ is the Frobenius of the maximal unramified extension $\mathbf{Q}_2(\mu_{2'})$ of \mathbf{Q}_2 . The topological generator 1 of $\hat{\mathbf{Z}}_2$ and the generator 1 of $\mathbf{Z}/2$ are the automorphisms of the field $\mathbf{Q}_2(\mu_2)$ determined by $\varepsilon \mapsto \varepsilon^5$ and $\varepsilon \mapsto \varepsilon^{-1}$ resp. for all $\varepsilon \in \mu_2$ (see e.g. [H], § 4, 5). We shall denote these automorphisms by σ_5 and σ_{-1} resp.

From (3) we get a "primary decomposition"

$$(4) \quad \mathcal{G}(\mathbf{Q}_2^c/\mathbf{Q}_2) \cong \prod_{p \neq 2} \hat{\mathbf{Z}}_p \times (\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2)$$

since $\hat{\mathbf{Z}} \cong \prod \hat{\mathbf{Z}}_p$. Since $\mathcal{G}(\mathbf{Q}_2^c/K)$ is an open subgroup, one can show that the isomorphism implied in (4) restricts to an isomorphism

$$\mathcal{G}(\mathbf{Q}_2^c/K) \cong \prod_{p \neq 2} k_p \hat{\mathbf{Z}}_p \times C_K \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2 \times D_K$$

where C_K is a $\hat{\mathbf{Z}}_2$ -submodule of finite index in the component $\hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_2 \times \mathbf{Z}/2$ of (4), k_p is an integer (or a power of p) = to 1 for almost all p , and D_K is either the trivial group or $\langle \sigma_{-1} \rangle$.

We now begin the proof of (2). Suppose first that $\sigma_{-1} \notin \mathcal{G}(\mathbf{Q}_2^c/K)$, i.e. that $\mathcal{G}(\mathbf{Q}_2^c/K) \cong \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}_2$. It suffices to show that $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$ is trivial. Let K_{nr} be the unramified closure of K in \mathbf{Q}_2^c . Then $K_{nr} = K(\mu_{2^r})$ and $\mathcal{G}(K_{nr}/K) \cong \hat{\mathbf{Z}}$.

Let C_n denote the cyclic group of order n .

LEMMA. Suppose C_{2^k} operates faithfully on C_{2^h} . Then $H^n(C_{2^k}, C_{2^h}) = 1$ for all $n \geq 1$ except in one case: $k = 1$ and the non-trivial automorphism in C_2 inverts the elements of C_{2^h} (i.e. " $C_{2^k} = \langle \sigma_{-1} \rangle$ ").

This is a well-documented fact, although perhaps not exactly in this form (see e.g. [N], 4.8, or the proof of Lemma 2, [L]). By the Herbrand theory for the cohomology of cyclic groups (see e.g. [S], ch. VIII, § 4), it suffices to show that $\hat{H}^0(C_{2^k}, C_{2^h}) = 1$, i.e. that every fixed element is a norm. There is generator of C_{2^k} which acts by raising the elements of C_{2^h} to either the power $5^{2^{h-k-2}}$, or possibly the power -5 if $k = h-2$ (again [H], § 4, 5). Then a straightforward calculation leads to the desired result (one uses the fact that $2^{r+2} \parallel (5^{2^r} - 1)$ for all $r \geq 0$). \square

Since

$$\mathbf{Q}_2^c = K_{nr}(\mu_2), H^n(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = \varinjlim H^n(\mathcal{G}(L/K_{nr}), \mu(L)_2)$$

where L runs over the fields $K_{nr}(\varepsilon_{2^h})$, and so is trivial by the lemma for $n \geq 1$. Thus the inflation-restriction sequence (p. 126, [C-F])

$$1 \rightarrow H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2) \rightarrow H^2(\mathcal{G}(\mathbf{Q}_2^c/K_{nr}), \mu_2) = 1$$

is exact whence the inflation is an isomorphism. But $\mathcal{G}(K_{nr}/K) = \hat{\mathbf{Z}}$ has cohomological dimension 1, so $H^2(\mathcal{G}(K_{nr}/K), \mu(K_{nr})_2)$ is 1, hence $H^2(\mathcal{G}(\mathbf{Q}_2^c/K), \mu_2)$ is also 1 as desired. (I am grateful to U. Jannsen for the foregoing proof).

We now assume that $\sigma_{-1} \in \mathcal{G}(\mathbf{Q}_2^c/K)$. This part of the proof is derived from pp. 540-542 in [J] and pp. 467-468, [L]. (F. Lorenz has asked me to point out that the proof on pp. 465-466 of the latter paper is incomplete — one must show that ρ is the identity on k .)

LEMMA 1. $K(\varepsilon_4)/K$ is ramified of degree 2.

Proof. It is clear that the extension is of degree 2. Suppose it is unramified. Let q be the number of elements in the residue class field of

$K(\varepsilon_4)$. Then $K(\varepsilon_4) = K(\varepsilon_{q-1})$. But $K(\varepsilon_{q-1})$ is left element-wise fixed by σ_{-1} , which contradicts the fact that ε_4 is *not* left fixed. \square

Let h be the smallest integer ≥ 2 such that there is an odd integer m with the property that $L = \mathbf{Q}_2(\varepsilon_{2^h}, \varepsilon_m)$ contains K . By replacing m by a suitable multiple, we can suppose that the residue class degree of L/K

$$f(L/K) \equiv 0 \pmod{2^h}.$$

Let \mathcal{G} be the Galois group of this extension. We shall construct a Schur class of K using L/K . For this we use the following very useful lemma. Let G be a finite abelian group, written as the direct sum of cyclic subgroups:

$$G = C_1 \oplus C_2 \oplus \dots \oplus C_r,$$

where each C_i is of order c_i with generator σ_i . Let A be a G -module, written multiplicatively. Define the operators Δ_{σ_i} and N_{σ_i} on A by

$$\Delta_{\sigma_i} a = a^{\sigma_i^{-1}}, N_{\sigma_i} a = a^{1 + \sigma_i + \sigma_i^2 + \dots + \sigma_i^{c_i - 1}}.$$

LEMMA. Let γ be a cohomology class in $H^2(G, A)$, and let f be a normalized cocycle in γ . Then the elements

$$\begin{aligned} a_i &= f(\sigma_i, \sigma_i) f(\sigma_i^2, \sigma_i) \dots f(\sigma_i^{c_i - 1}, \sigma_i), \\ a_{ij} &= f(\sigma_i, \sigma_j) / f(\sigma_j, \sigma_i) \quad (i \neq j) \end{aligned} \tag{5}$$

satisfy the following relations:

$$\Delta_{\sigma_i} a_j = \begin{cases} 1 & \text{if } i = j \\ N_{\sigma_j} a_{ij} & \text{if } i \neq j \end{cases}, \tag{6}$$

$$a_{ij} a_{ji} = 1 \quad (i \neq j), \quad \Delta_{\sigma_i} a_{jk} \cdot \Delta_{\sigma_j} a_{ki} \cdot \Delta_{\sigma_k} a_{ij} = 1 \quad (i, j, k \text{ distinct}).$$

Conversely if we have elements a_i and a_{ij} in A satisfying (6), then there is a uniquely determined cohomology class γ in $H^2(G, A)$ and a normalized cocycle f in γ bearing the relationship (5) to the a_i and a_{ij} .

Proof. This is just a restatement of the abelian case of [Z], III, § 8, Theorem 22, in terms of cocycles. See also [Y], pp. 15-19. \square

We now apply this to the situation at hand: $G = \mathcal{G}$ and $A = \mu(L)_2 = \langle \varepsilon_{2^h} \rangle$. First of all we note that the restriction σ_1 of σ_{-1} to L is a non-trivial element of \mathcal{G} , and that the minimality of h implies that

$K(\varepsilon_4, \varepsilon_m) = K(\varepsilon_{2^h}, \varepsilon_m) = L$ (see e.g. Lemma 3.3, [J]). Since $K(\varepsilon_4)/K$ is ramified and $K(\varepsilon_m)$ is unramified (because m is odd), \mathcal{G} is the direct product of the Galois groups $\langle \sigma_1 \rangle$ of $L/K(\varepsilon_m)$ and $\langle \sigma_2 \rangle$ (say) of $L/K(\varepsilon_4)$ of orders 2 and f respectively. We now choose $a_1 = 1 = a_2$ and $a_{12} = \varepsilon_{2^h} = a_{21}^{-1}$. Then $N_{\sigma_1} a_{21} = \varepsilon_{2^h}^{-1} \varepsilon_{2^h} = 1$ and $N_{\sigma_2} a_{12} = \varepsilon_{2^h}^s$ where, if $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}^r$,

$$s = 1 + r + r^2 + \dots + r^{f-1}.$$

Since $\sigma_2(\varepsilon_4) = \varepsilon_4$, we have $r \equiv 1 \pmod{4}$.

Claim: $s \equiv 0 \pmod{2^h}$. If $\sigma_2(\varepsilon_{2^h}) = \varepsilon_{2^h}$, we choose $r = 1$; then $s = f \equiv 0 \pmod{2^h}$.

Suppose then that $\sigma_2(\varepsilon_{2^h}) \neq \varepsilon_{2^h}$, and write $s = (r^f - 1)/(r - 1)$. Now $r = 1 + 2^k a$ where $h > k \geq 2$ and a is odd. By induction $r^{2^i} = 1 + 2^{k+i} a_i$ (a_i an odd integer) for all $i \geq 0$, whence the claim.

It follows of course that $N_{\sigma_2} a_{12} = 1$. Therefore the above lemma provides a 2-cocycle f with coefficients in $\langle \varepsilon_{2^h} \rangle$. We now consider it to have coefficients in L^* and so its cohomology class $\gamma = [f]$ is a Schur class in $B(K)$. We shall show that this class is non-trivial, which will finish the proof of the theorem. This will be effected by showing that γ is the inflation of a non-trivial Brauer class arising from the extension $K(\varepsilon_m)/K$ — this latter class will not arise from a cyclotomic cocycle but this of course does not matter.

We shall use the crossed-product algebra $A = (L/K, f)$ in order to carry this out. As a vector space over L it has a basis $u_1^i u_2^j$ where $0 \leq i < 2$ and $0 \leq j < f$, with $u_1^2 = 1 = u_2^f$ and $u_1 u_2 u_1^{-1} u_2^{-1} = \varepsilon_{2^h}$. We replace u_2 by $u'_2 = \pi u_2$ where $\pi = \varepsilon_4(1 - \varepsilon_{2^h})$. The new parameters are

$$a'_1 = u_1^2 = 1, \quad a'_{12} = u_1 u'_2 u_1^{-1} u'_2^{-1} = 1, \quad a'_2 = u'^f_2 = N_{\sigma_2} \pi.$$

By (6), $\Delta_{\sigma_1} a'_2 = N_{\sigma_2} a'_{12} = 1$ and $\Delta_{\sigma_2} a'_2 = 1$, so $N_{\sigma_2} \pi \in K$. Since u_1 and u'_2 commute with each other, it follows easily that

$$A = (K(\varepsilon_4)/K, 1) \otimes (K(\varepsilon_m)/K, N_{\sigma_2} \pi).$$

The first of these crossed-product algebras is clearly split but the second is *not* split: π is a prime element of L , so $N_{\sigma_2} \pi$ has order (valuation) f in $K(\varepsilon_4)$ ($L/K(\varepsilon_4)$ is unramified), hence order $1/2 f$ in K ($K(\varepsilon_4)/K$ is ramified); but the (non-zero) norms in K from $K(\varepsilon_m)$ are exactly the elements whose order is a multiple of f (since the extension is unramified of degree f). Thus A is not split, as desired. \square