

## §4. Quillen's Theorem for Loop Groups

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and  $I$  is a proper subset of  $\tilde{S}$ , then  $\tilde{W}_I$  is finite. This is obvious since the elements of  $I$  have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take  $A_s = \tilde{G}_s$ . We have  $\tilde{G}_s \tilde{B} = \tilde{G}_{c,s} \tilde{B} = \tilde{B}$   $U_s \tilde{B} = P_s$ . In particular  $P_s / \tilde{B} = \tilde{G}_s / (\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbb{C}P^1$ , which also proves Axioms (2.20) and (2.21).  $\square$

(3.2) COROLLARY.  $\Omega_{alg} G$  is a CW-complex with cells of even dimension, indexed by  $\text{Hom}(S^1, T)$ . The Poincaré series for its integral homology is  $\sum_{\lambda \in \text{Hom}(S^1, T)} t^{2\bar{l}(\lambda)}$ , where  $\bar{l}(\lambda)$  is the minimal length accruing in  $\lambda W$ . Identifying  $\text{Hom}(S^1, T)$  with  $\tilde{W}^S$ , the closure relations on the cells are given by the Bruhat order on  $\tilde{W}^S$ .  $\square$

*Remark.* An explicit formula for  $\bar{l}(\lambda)$  is given in [16], Prop. 1.25:  $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$ .

We will also need the “Iwasawa decomposition” (see [17], [27], [29]):

(3.3) THEOREM.  $\tilde{G}_C = \Omega_{alg} G \times P$ .  $\square$

*Remark.* Note that (3.3) shows that the associated building, which we will be denoted simply by  $\mathcal{B}_G$ , is a quotient of  $L_{alg} G / T \times \Delta$ . The equivalence relation is then  $(f_1 T, X) \sim (f_2 T, X)$  if  $X \in \Delta_I$  and  $f_1 = f_2 \bmod LG \cap P_I$ .

#### § 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. Let  $G$  be a compact Lie group. Then the inclusion  $\Omega_{alg} G \rightarrow \Omega G$  is a homotopy equivalence.

If  $G$  is simply connected, let  $\mathcal{B}_G$  denote the topological building associated to the algebraic loop group  $L_{alg} G_C$  as in § 2.

(4.2) THEOREM (Quillen).  $\Omega_{alg} G$  acts freely on  $\mathcal{B}_G$ , with orbit space  $G$ .

*Proof of (4.1).* It is easy to reduce to the case when  $G$  is simply connected. Since  $B_G$  is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that  $\Omega_{alg} G \rightarrow \Omega G$  is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence.  $\square$

Since  $G$  is a product of simple groups (as is  $G_{\mathbb{C}}$ ), it is very easy to reduce to the case when  $G$  is simple. For the rest of this section, then, we assume  $G$  is simple and simply-connected, of rank  $l$ .

To prove 4.2, we introduce Quillen's space of special paths  $\mathcal{S}_G$ : this is the space of all paths  $[0, 1] \rightarrow G$  of the form  $f(e^{2\pi it}) \exp tX$ , where  $f \in \Omega_{\text{alg}} G$  and  $X \in \mathfrak{g}$ .  $\mathcal{S}_G$  is topologized as a quotient of  $\Omega_{\text{alg}} G \times \mathfrak{g}$ . Note that  $L_{\text{alg}} G$  acts on  $\mathcal{S}_G$  by  $h \cdot (f \exp tX) = hf \exp tXh(1)^{-1}$ . The following key lemma, whose proof is deferred, also helps to explain the significance of the parabolic subgroups  $P_I$ .

(4.3) LEMMA. Suppose  $X \in \mathring{\Delta}_I$ , then the isotropy group of  $\exp tX$  is  $L_{\text{alg}} G \cap P_I$ .

(4.4) THEOREM (Quillen).  $\mathcal{S}_G$  is  $L_{\text{alg}} G$ -equivariantly homeomorphic to the building  $\mathcal{B}_G$ .

*Proof.* The action map  $\varphi: L_{\text{alg}} G \times \Delta \rightarrow \mathcal{S}_G$  given by

$$\varphi(f, X) = f \exp tX f(1)^{-1}$$

is surjective by Theorem 1.1. If  $\varphi(f_1, X_1) = \varphi(f_2, X_2)$ , then (evaluating at  $t=1$ )  $\exp X_1$  and  $\exp X_2$  are conjugate in  $G$ , so  $X_1 = X_2$  by Theorem 1.3. We then have  $\varphi(f_1, X) = \varphi(f_2, X)$  if and only if  $f_1 = f_2$  mod the isotropy group of  $\exp tX$ . Hence, by (4.3),  $\varphi$  factors through the desired homeomorphism  $\mathcal{B}_G \rightarrow \mathcal{S}_G$ .  $\square$

*Remark.* Here we have used the Iwasawa decomposition (3.3) to identify  $\mathcal{B}_G = (\tilde{G}_{\mathbb{C}}/\tilde{B} \times \Delta)/\sim$  with  $(L_{\text{alg}} G/T \times \Delta)/\sim$ .

(4.5) LEMMA.  $L_{\text{alg}} G \cap P_I$  is generated by  $T$  and the subgroups  $\tilde{G}_i, i \in I$ .

*Proof.* We have  $P_I = \tilde{B}W_I\tilde{B}$ . By the Steinberg lemma (2.9), each  $\tilde{B}w\tilde{B} (w \in W_I)$  has the form  $XB$ , where  $X$  is a product of the  $\tilde{G}_i$ . Since  $L_{\text{alg}} G \cap XB = XT$ , the lemma follows.  $\square$

*Proof of 4.2.* The action of  $\Omega_{\text{alg}} G$  on  $\mathcal{S}_G$  is clearly free. By (4.4), the same is true for  $\mathcal{B}_G$ . Now consider the orbit space  $\mathcal{B}_G/\Omega_{\text{alg}} G$ . Since  $\mathcal{B}_G = (L_{\text{alg}} G/T \times \Delta)/\sim = (\Omega_{\text{alg}} G \times G/T \times \Delta)/\sim$ , the orbit space is a quotient of  $G/T \times \Delta$ . The equivalence relation is given by  $(g_1 T, X) \sim (g_2 T, X)$  if  $X \in \mathring{\Delta}_I$  and  $g_2 = f g_1 p$  with  $f \in \Omega_{\text{alg}} G, p \in P_I$ . In fact  $p \in LG \cap P_I$ . Now let  $\bar{G}_I = e(LG \cap P_I)$ , where  $e$  is evaluation at  $z = 1$ . Then  $(g_1 T, X) \sim (g_2 T, X)$  if and only if  $g_1 = g_2 \text{ mod } \bar{G}_I$ . For if  $g_2 = f g_1 p$  as above, then  $\bar{G}_I = e(L_{\text{alg}} G \cap P_I)$ , where  $e$  is evaluation at  $z = 1$ . Then  $(g_1 T, X) \sim (g_2 T, X)$  if and only if  $g_1 = g_2 \text{ mod } \bar{G}_I$ . For if  $g_2 = f g_1 p$  as above, then

$g_2 = f g_1 p(1)$ , and conversely if  $g_2 = g_1 p(1)$ , then  $g_2 = f g_1 p$ , where  $f = g_2 p^{-1} g_1^{-1}$ . But by (4.5),  $\bar{G}_I = G_I$  (see § 1). In other words, the equivalence relation here coincides with the classical relation of Theorem 1.5, which has quotient  $G$ .  $\square$

*Proof of 4.3.* Fix  $X \in \dot{\Delta}_I$ . We first show that  $L_{alg}G \cap P_I$  fixes  $\exp tX$  in  $\mathcal{S}_G$ . By (4.5) it is enough to show that each  $\tilde{G}_i (i \in I)$  fixes

$$\exp tX : f(e^{2\pi it}) \exp tX f(1)^{-1} = \exp tX.$$

If  $i \neq 0$ ,  $\tilde{G}_i = G_i$  is a subgroup of the constant loops, so  $f$  is a constant  $g \in G_i$ . The desired equation is then equivalent to  $g \cdot X = X$  (recall that  $g \cdot X = \text{Ad}(g)X$ ). But since  $i \neq 0$ ,  $\alpha_i(X) = 0$ , so this is true by definition. Now suppose  $i = 0$ , so that  $X$  lies on the outer wall:  $\alpha_0(X) = 1$ . Then  $X = \frac{1}{2} \alpha_0^* + Y$ , where  $\alpha_0^* = 2\alpha_0/\alpha_0 \cdot \alpha_0$  is the coroot of  $\alpha_0$  and  $\alpha_0(Y) = 0$ .

The equation we want can be written ( $f \in \tilde{G}_0$ ):

$$f(e^{2\pi it}) = \exp tX f(1) \exp -tX$$

Since  $f(1) \in G_0$ ,  $f(1) \cdot Y = Y$ , and our equation simplifies to

$$f(e^{2\pi it}) = \exp \left( \frac{1}{2} t \alpha_0^* \right) f(1) \exp \left( -\frac{1}{2} t \alpha_0^* \right)$$

Note this is now an equation in the path space of  $G_0$ . Identifying  $G_0$  with  $SU(2)$ , it can be written

$$\begin{pmatrix} a & be^{2\pi it} \\ ce^{-2\pi it} & d \end{pmatrix} = \begin{pmatrix} e^{\pi it} & 0 \\ 0 & e^{-\pi it} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\pi it} & 0 \\ 0 & e^{\pi it} \end{pmatrix}$$

Where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ . This last equation is obviously correct, and we conclude that  $L_{alg}G \cap P_I$  fixes  $\exp tX$ .

Conversely, suppose

$$f \exp tX f(1)^{-1} = \exp tX, \quad \text{or} \quad f = \exp tX f(1) \exp (-tX).$$

Then  $f(1) \in C_G \exp X = G_I$ , and hence  $f(1) = h(1)$  for some  $h \in L_{alg}G \cap P_I$ . But then  $h = \exp tX h(1) \exp -tX = f$ .

A useful fact that follows from all this is:

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism  $L_{alg}G \cap P_I \cong G_I$ . In particular,  $L_{alg}G \cap P_I$  is a compact Lie group.*

*Proof.* We have seen that  $e$  maps  $L_{alg}G \cap P_I$  onto  $G_I$ . The kernel is  $\Omega_{alg}G \cap P_I$ . But  $\Omega_{alg}G$  acts freely on  $\mathcal{S}_G$ , and  $L_{alg}G \cap P_I$  fixes  $\Delta_I$ , so  $\Omega_{alg}G \cap P_I = \{1\}$ .

*Remark.* As always,  $I$  is a proper subset of  $\tilde{S}$  in (4.6). Of course (4.6) also depends on our assumption that  $G$  is simple. Its discrete analogue is the fact that  $W_I$  is finite if  $\tilde{W}$  is irreducible. (It may be helpful to consider the “discrete” versions of all the results of this section. For example, the discrete version of “ $\Omega_{alg}G$  acts freely on  $B_G$ ” is “the coroot lattice  $\text{Hom}(S^1, T)$  acts freely on  $t$  (the foundation of  $\mathcal{B}_G$ )”; of course the latter assertion is trivial).

Note that we have shown that  $\mathcal{S}_G/\Omega_{alg}G = G$ , and in fact the orbit map  $\mathcal{S}_G \rightarrow G$  is given by evaluation at  $t = 1$ . This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose  $X, Y \in \mathfrak{g}$  and  $\exp X = \exp Y$ . Then  $\exp tX = f(e^{2\pi it}) \exp tY$  for some  $f \in \Omega_{alg}G$ .*  $\square$

It is not hard to prove this directly—for example, it is enough to prove it for  $G = U(n)$ . Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when  $G$  is simple and simply-connected. Using (1.3), one can easily reduce further to the case  $X \in \dot{\Delta}_I$ ,  $Y = g \cdot X$  for some  $g \in G$ . Then  $g \in C_G \exp X = G_I$ , so  $g = h(1)$  with  $h \in L_{alg}G \cap P_I$ . Let  $h = \exp tX g \exp -tX$ ; then  $h \in L_{alg}G$  and  $f = hh(1)^{-1}$  is the desired loop.

## § 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that  $G$  is simple and simply-connected. If  $\sigma$  is an involution on  $G$  with fixed group  $K$ , as usual, then  $K$  is connected and  $G/K$  is simply-connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space  $\Omega(G/K)$  is homotopy equivalent to the space of paths in  $G$  that start at the identity and end in  $K$ . Now consider the involution  $\tau$  on  $\Omega G$  given by  $\tau(f)(z) = \sigma(f(\bar{z}))$ . The fixed group  $(\Omega G)^\tau$  is clearly homeomorphic to our space of paths, since  $f \in (\Omega G)^\tau$  implies  $f(-1) \in K$ .