# §2. Topological Buildings

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

$y = s_1 \cdots s_k (s_i \in S)$ and $x$ has a reduced decomposition obtained by deleting some subset of the $s_i$'s occuring in $y$. (For a very nice account of these related matters, see [14]). If $W$ is finite, $W$ has a unique element $w_0$ of maximal length, we define the length of $W$ to be $l(w_0)$.

## § 2. Topological Buildings

A *Tits system* $(G, B, N, S)$ consists of a group $G$, subgroups $B$ and $N$, and a set $S$, which satisfy the following axioms:

(2.1) $B \cap N$ is normal in $N$, and $S$ is a set of involutions generating $W \equiv N/B \cap N$,

(2.2) $B$ and $N$ generate $G$,

(2.3) If $s \in S$, $sBs \neq B$,

(2.4) if $s \in S$, $w \in W$, then $sBw \leqslant BwB \cup BswB$.

(The use of expressions such as $sBw$ is a standard abuse of notation).

*Example.* Let $G$ be a reductive algebraic group over an algebraically closed field (e.g., $GL(n, \mathbf{C})$), let $B$ be a Borel subgroup (e.g. upper triangular matrices), and let $N$ be the normalizer of a maximal torus (that lies in $B$). This data determines a set $S$ of simple reflections generating the Weyl group $W$ (e.g., the usual generators $s_1, \ldots, s_{n-1}$ of $\Sigma_n$). Then one of the main results in the structure theory of reductive groups is that $(G, B, N, S)$ is a Tits system (see for example [15]).

Throughout this paper we will assume that the set $S$ is finite; its cardinality $l$ is the *rank* of the system.

We next list some of the important properties of a Tits system.

(2.5) (Bruhat Decomposition) $G = \coprod_{w \in W} BwB$ (disjoint union),

(2.6) $(W, S)$ is a Coxeter system.

A subgroup $P$ of $G$ is *parabolic* if it contains a conjugate of $B$. In particular if $I \subseteq S$, the subgroup $P_I$ generated by $B$ and $I$ is parabolic.

(2.7) (a) The parabolic subgroups containing $B$ are precisely the $P_I$, $I \subseteq S$. No two of these are conjugate; in particular there are exactly $2^l$ such subgroups, which form a lattice isomorphic to the lattice of subsets of $S$.

(b) $P_I = BW_IB$

(c) Every parabolic $P$ is self-normalizing: $N_G P = P$.

(2.8) (Bruhat decomposition, general version) $G = \coprod_{w \in W_I \backslash W / W_J} P_I w P_J$ (disjoint union).

The next result, which we will refer to as the *Steinberg Lemma*, is somewhat technical; however it is not hard to prove and is extremely useful. It is a mild generalization of Theorem 15 of [32] and Proposition 3.1 of [19].

(2.9) Let $I \subseteq S$ and suppose $w$ is the unique element of minimal length of $wW_I$. Suppose $w = w_1 \dots w_k$ where $l(w) = l(w_1) + \dots + l(w_k)$. Then

(a) If $Y_i$ is any subset of $Bw_iB$ such that $Y_i \to Bw_iB/B$ is bijective (resp. surjective) $(1 \leqslant i \leqslant k)$, then $Y_1 \times Y_2 \dots \times Y_k \to BwP_I/P_I$ is bijective (resp. surjective).

(b) Suppose $w_i \in S$, $1 \leqslant i \leqslant k$ i.e., $w_1 \dots w_k$ is a reduced decomposition of $w$). Let $Z_i$, $1 \leqslant i \leqslant k$, be any subset containing 1 of $P_{w_i}$ such that $Z_i \to P_{w_i}/B$ is surjective. Then the image of $Z_1 \times \cdots Z_k \to G/P_I$ is $\coprod_{x \leqslant w} BxP_I/P_I$.

The maps in (a), (b) are the obvious multiplication/projection maps. Part b refers to the Bruhat order on $W^I$.

(2.10) *Remark.* The Tits system of a reductive algebraic group has several additional features: $B = HU$, where $H$ is a maximal torus and $U$ is a normal unipotent subgroup, $U$ in turn is described in terms of its root subgroups, and there is an "opposite" Borel subgroup $B^-$ such that $B \cap B^- = H$. This additional structure can also be axiomatized in an elegant way, leading to the "refined" Tits system of Kac and Peterson [19]. One then obtains, for example, the *Birkhoff decomposition* $G = \coprod_{w \in W} B^- wB$ as a consequence of the axioms.

We now define a *topological* Tits system to be a Tits system such that $G$ is a topological group, $B$ and $N$ are closed subgroups, and $W$ is discrete (i.e. $N \cap B$ is an open subgroup of $N$). We will usually also assume (for reasons which will be apparent shortly):

(2.11) *Axiom.* If $I$ is a proper subset of $S$, $W_I$ is finite.

This axiom is satisfied if $W$ is an irreducible affine Weyl group, or finite. To get any interesting results some further axiom seems necessary. One direction is considered in [11], where the groups in question are algebraic groups over local fields, with the valuation topology. Here, with loop groups in mind, the following axiom seems efficient:

(2.12) *Axiom.* For each $s \in S$ there is a subset $A_s$ of $P_s$ such that (a) $A_sB = P_s$, (b) $A_s$ is compact and contains 1, and (c) $A_s = \overline{A_s \cap BsB}$. This axiom is motivated by Steinberg's approach [32].

(2.13) PROPOSITION. *Let* $(G, B, N, S)$ *be a topological Tits system satisfying* (2.12). *Then*

(a) $\overline{BwB} = \coprod_{x \leqslant w} BxB (w \in W)$. *More generally if* $I \leqslant S$, *and* $w \in W^I$, $\overline{BwP_I} = \coprod_{x \leqslant w} BxP_I$ *(here* $x \in W^I$*)*,

(b) *B-orbits in* $G/P_I$ *are locally closed,*

(c) *If* $W$ *satisfies (2.11), parabolic subgroups are closed.*

*Proof.* First we show $P_s = \overline{BsB}$. Since $P_s = A_sB$, with $A_s$ compact and $B$ closed, $P_s$ is closed, so $P_s \geqslant \overline{BsB}$. But also $B \subset P_s = A_sB \subset \overline{BsB}$, which proves our claim. Part (a) now follows easily from the Steinberg lemma: Let $M_w = \coprod_{x \leqslant w} BxP_I$, and let $w = s_1 \cdots s_k$ be a reduced decomposition. Then $M_w = A_1 \cdots A_k P_I$ and hence is closed. Next, suppose $x \leqslant w$; we must show $BxB \leqslant \overline{BwB}$. It is enough to consider the case when $X$ has a reduced decomposition $x = s_1 \cdots \hat{s}_i \cdots s_k$ (omit $s_i$). Then

$$BxP_I = A'_1 \cdots A'_{i-1} A'_{i+1} \cdots A'_k P_I \leqslant A'_1 \cdots A'_{i-1} \bar{A}_i \cdots A'_k P_I \leqslant \overline{BwP_I}$$

(since $1 \in A_i$), where $A'_i = A_i \cap Bs_iB$. This proves (a). Part (b) is immediate since the complement of $BwP_I$ in its closure is a finite union of sets of the form $M_x$, hence is closed. Since $P_I = BW_IB$, (c) is also immediate from (a) if $W_I$ is finite. □

From now on we will assume 2.11 and 2.12. The homogeneous spaces $G/P_I$ will be called *flag spaces*. The $B$-orbits $E_w = BwP_I/P_I$ are *Schubert strata* and the compact subspaces $\overline{E_W}$ are *Schubert subspaces*.

We next consider the *building* $\mathscr{B}_G$ associated to a topological Tits system $(G, B, N, S)$. (The notation is ambiguous—indeed in the case of loop groups, $G$ will support two natural but totally different Tits system. However the system we have in mind will be clear from the context.) In the discrete case, $\mathscr{B}_G$ is usually defined as the following simplicial complex. The vertices are the maximal (proper) parabolics, and $P_1 \cdots P_k$ span a simplex if $\bigcap_{i=1}^{k} P_i$ contains a conjugate of $B$. In general it is convenient to reinterpret this definition as follows: first of all, by definition every parabolic $P$ is conjugate to a unique $P_I$; we say that $P$ has type $I$. Thus the maximal parabolics are the parabolics of type $[s]$, where $[s] = S - \{s\}$. More generally the $k$-simplices correspond to the parabolics of type $I$, where $|I| = l - k - 1$. Thus the simplices all have dimension $\leqslant l - 1$, with the $l - 1$ simplices corresponding to the conjugates of $B$. Furthermore, in view of 2.7 (c), the set of parabolics of type $I$ is canonically identified with $G/P_I - xP_I$ corresponding to $xP_Ix^{-1}$. One can casily check that with this interpretation, a simplex $xP_I$ is a face of a simplex $yP_J$ if and only if $I \supset J$ and $xP_I = yP_I$. In particular, every simplex is a face of some $l-1$ simplex. Hence, as a set, $B_G$ can be identified with $G/B \times \Delta/\sim$, where $\Delta$ is the $l-1$ simplex with vertex set $S$, and $(g_1B, X_1) \sim (g_2B, X_2)$ if $X_1 = X = X_2, X \in \Delta_I$, and $g_1P_I = g_2P_I$. (Here $\Delta_I$ is the face of $\Delta$ corresponding to $I \leqslant S$.) We will therefore *define* the building $\mathscr{B}_G$ associated to the topological Tits system $(G, B, N, S)$ to be $G/B \times \Delta$ modulo this equivalence relation, with the quotient topology.

*Remark.* Another way of expressing this is as follows: Let $C$ be the category defined by the poset of proper subsets of $S$ (including the empty set). We have a functor from $C$ to topological spaces given by $I \mapsto G/P_I$. Then $\mathscr{B}_G$ is precisely the homotopy colimit of this diagram of spaces, in the sense of [8], p. 327 ff.

(2.14) PROPOSITION. *The equivalence relation on* $G/B \times \Delta^{l-1}$ *is generated by the relations* $(g_1B, X) \sim (g_2B, X)$ *if* $X$ *lies on the wall* $\Delta_s$ *and* $g_1P_s = g_2P_s$.

*Proof.* In the usual language, (2.14) is the assertion that any two chambers are linked by a "gallery". (See e.g. [11], appendix.) Since the action of $G$ on $G/B$ induces a well defined action on $\mathscr{B}_G$, we are reduced to showing that if $(B, X) \sim (gB, X)$—i.e. $X \in \Delta_I$ and $g \in P_I$—then $(B, X)$ and $(gB, X)$ are linked by a sequence of relations of the stated type. But $gB = bwB$ with $w \in W_I$; hence if $w = s_1 \cdots s_k$ is a reduced decomposition, the elements $(B, X), (bs_1B, X), (bs_1s_2B, X), \ldots (bwB, X)$ provide the desired sequence. □

Note that the set $\Delta$ is a fundamental domain for the action of $G$ on $\mathscr{B}_G$. On the other hand, it is easy to check that the closed subspace $\mathscr{B}_W$ consisting of the pairs $(wB, X), w \in W$, is a fundamental domain for the $B$ action. (The point is that if $bw_1P_I = w_2P_I$, then $w_1P_I = w_2P_I$, by the Bruhat decomposition.) This space $\mathscr{B}_W$, which we will call the *foundation* of the building, is a simplicial complex since $W$ is discrete. Since it will turn out that $\mathscr{B}_G$ is in a sense a "thickening" of the foundation, the following well known description of $\mathscr{B}_W$ may be of interest.

(2.15) PROPOSITION. *Suppose* $\Phi$ *is an irreducible root system in the Euclidean space* $V$. *Then*

(a) *If* $W$ *is the affine Weyl group associated to* $\Phi$, *then* $\mathcal{B}_W$ *is isomorphic as a simplicial* $W$-*complex to* $V$ *(triangulated by the hyperplanes of* $\Phi$*).*

(b) *If* $W$ *is the Weyl group of* $\Phi$, $\mathcal{B}_W$ *is isomorphic as simplicial* $W$-*complex to the unit sphere of* $V$, *triangulated by the Weyl chambers. More precisely,* $\mathcal{B}_W$ *can be identified with the* $W$ *orbit of the outer wall of the Cartan simplex.*

*Proof.* For (a), map $W \times \Delta \xrightarrow{\varphi} V$ by identifying $\Delta$ with the Cartan simplex in $V$ and using the action map. Then $\varphi$ is onto (1.1) and furthermore $\varphi(w_1, x) = \varphi(w_2, X_2)$ if and only if $X_1 = X = X_2, X \in \Delta_I$, and $w_1 = w_2$ modulo the isotropy group of $X$. But this isotropy group is precisely $W_I$ (1.2), so $\varphi$ factors through the desired isomorphism $\mathcal{B}_W \to V$. The proof of (b) is similar. □

We now come to the main result of this section. Filter $G/B$ by $F_k(G/B) = \coprod_{l(w) \leqslant k} E_w$. Similarly, $\mathcal{B}_G$ is filtered by $F_k(\mathcal{B}_G) = F_k(G/B) \times \Delta/\sim$.

(2.16) THEOREM. *Let* $(G, B, N, S)$ *be a topological Tits system which either is discrete or satisfies (2.11) and (2.12). Assume also that the inclusions* $F_k(B_G) \subset F_{k+1}(B_G)$ *are cofibrations. Then*

(a) *If* $W$ *is infinite,* $\mathcal{B}_G$ *is contractible.*

(b) *If* $W$ *is finite of length* $r$, $\mathcal{B}_G$ *is homotopy equivalent to the* $(l-1)$ *st suspension* $S^{l-1} \wedge (F_r(G/B))/F_{r-1}(G/B))$.

*Remark.* If $G$ is discrete, $F_k\mathcal{B}_G$ is a subcomplex of the simplicial complex $\mathcal{B}_G$, so the cofibration hypothesis is automatically satisfied. Furthermore if $W$ is finite the smash product in (b) is just a wedge of $|F_rG/B - F_{r-1}G/B|$ $(l-1)$-spheres. This case is due to Solomon and Tits; cf. [11].

*Proof of (2.16).* Let $X_k$ denote $F_k\mathcal{B}_G/F_{k-1}\mathcal{B}_G$, and let $X'_k = F_k(G/B)/F_{k-1}(G/B)$. Then we will show

(2.17) If $k$ is less than the length of $W$, $X_k$ is contractible. If $k = r$ = length of $W$, $X_k$ is homeomorphic to $(F_r(G/B)/F_{r-1}(G/B) \wedge S^{l-1})$.

If $W$ is infinite, it follows that $F_k\mathcal{B}_G$ is contractible for all $k$, and hence $\mathcal{B}_G$ is contractible. If $W$ is finite, part (b) of the theorem is also immediate.

To prove 2.17, first consider the quotient map $\pi: F_k(G/B) \times \Delta \to X_k$. In fact $\pi$ is merely collapsing a subspace to a point:

(2.18) Let $A_1 = (b_1 w_1 B, X_1)$, $A_2 = (b_2 w_2 B, X_2)$. If $\pi(A_1) = \pi(A_2)$, then either $A_1 = A_2$ or $\pi(A_1) = \pi(A_2) = *$ ($*$ is the basepoint $F_{k-1}B_G$).

For suppose $\pi(A_1) \neq *$, and $X_1 = \mathring{\Delta}_I$. Then $l(w_1) = k$ and $w_1 \in W^I$. This forces $X_1 = X_2$ and $w_1 = w_2 \bmod W_I$; hence $w_1 = w_2$ since $l(w_2) \leqslant k$ by assumption. Then $b_1 w_1 P_I = b_2 w_1 P_I$. But whenever $w \in W^I, b_1 w P_I = b_2 w P_I$ implies $b_1 w B = b_2 w B$ (easy exercise).

It now follows that $X_k = \bigvee_{l(w)=k} X_w$, where $X_w$ is the image of $\bar{E}_w \times \Delta$ in $X_k$, and to prove (2.17) we need only consider a fixed $X_w$. Let $X'_w = \bar{E}_w/(\bar{E}_w - E_w)$, and let $\Delta'$ be the subcomplex of $\Delta$ consisting of the walls $\Delta_s$ such that $l(ws) < l(w)$. Then (2.18) implies:

(2.19) $X_w = X'_w \wedge (\Delta/\Delta')$.

For $X_w$ is $\bar{E}_w \times \Delta$ modulo the subspace of points which are equivalent (in $\mathscr{B}_G$) to a point of lower filtration, namely, $\bar{E}_w \times \Delta' \cup \bar{E}_w - E_w \times \Delta$. It remains to identify $\Delta'$. Since $F_0\mathscr{B}_G = \Delta$ is contractible, we may assume $k \geqslant 1$; then $\Delta'$ is nonempty. If $k < l(W)$, then there is at least one $s \in S$ such that $l(ws) > l(w)$; hence $\Delta'$ is not the entire boundary of $\Delta$ and $\Delta/\Delta'$ is contractible. If $k = l(W)$, then $w$ is unique, $\Delta' =$ boundary of $\Delta$, and $\Delta/\Delta' = S^{l-1}$. This completes the proof of (2.17), and of the theorem. □

*Remark.* Our proof of Theorem 2.16 is an adaptation of the standard (discrete) proof to the topological setting. Much of the proof depends only on the Weyl group $W$, and indeed shows e.g. for $W$ infinite that the foundation of the building is contractible. In fact the deformation of $F_k(\mathscr{B}_W)$ into $F_{k-1}(\mathscr{B}_W)$ has the property that the isotropy group in $B$ of a point $X$ in $\mathscr{B}_W$ is an increasing function of time, and hence extends uniquely to a $B$-equivariant deformation of $F_k(B_G)$. In the discrete case this extension is automatically continuous, and shows that Theorem (2.16) holds $B$-equivariantly. (This was observed, (not for the first time) in [21], and has an interesting application concerning the Steinberg representation of a finite Chevalley group.) However this proof does not work in the topological case; simple counterexamples show that the extension will be discontinuous.

In many cases the Bruhat decomposition of $G/P$ is in fact a *CW* decomposition. The following axioms are convenient in this regard:

(2.20) *Axiom.* For each $s \in S$, the projection $P_s \to P_s/B$ has a local section.

(2.21) *Axiom.* For each $s \in S$, $P_s/B$ is homeomorphic to a sphere of positive dimension.

We then have:

(2.22) THEOREM. *Let* $(G, B, N, S)$ *be a topological Tits system satisfying axioms 2.11, 2.20 and 2.21. Let* $P \equiv P_I$ *be a parabolic subgroup,* $I \leqslant S$, *and give* $G/P$ *the compactly generated topology. Then*

(a) *Axiom 2.12 is satisfied.*

(b) *The Bruhat decomposition of* $G/P$ *is a* $CW$ *decomposition, and the closure relations on the cells are given by the Bruhat order on* $W^I$.

(c) *The building* $\mathscr{B}_G$ *satisfies the cofibration condition of Theorem 2.16.*

*Proof.* By assumption there are maps $D^{m(s)} \xrightarrow{\varphi_s} P_s/B$ such that $\varphi_s^{-1}(B) = \partial D^{m(s)}$ and $D^{m(s)}/\partial D^{m(s)} \to P_s/B$ is a homeomorphism. Furthermore $\varphi_s$ lifts to a map $\tilde{\varphi}_s\colon D^{m(s)} \to P_s$ with $1 \in \tilde{\varphi}_s(\partial D^{m(s)})$. Thus, in Axiom (2.12) we may take $A_s = \tilde{\phi}_s(\mathring{D}^{m(s)})$, proving (a). Since $P$ is closed (2.13c), $G/P$ is a Hausdorff space. If $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, the Steinberg lemma (2.9) shows that the multiplication map $D^{m(s_1)} \times \cdots \times D^{m(s_k)} \to \bar{E}_w$ (using $\tilde{\varphi}_{s_i}$) is a characteristic map for the cell $E_w$. The boundary of each cell is a finite union of cells of lower dimension by 2.13a, and $G/P$ has the weak topology by assumption. The closure relations also follow from (2.13). This proves (b). For (c) we observe that $\mathscr{B}_G$ (with the compactly generated topology) is itself a $CW$-complex, and the filtrations $F_k\mathscr{B}_G$ are subcomplexes: Indeed if we regard $\mathscr{B}_G$ as a quotient space of $\coprod_{I \leqslant S}(G/P_I \times \Delta_I)$, it is clear that there is one cell for each $I < S$ and $w \in W^I$. $\square$

If $G, P_I$ are as in the above theorem, and $w \in W^I$ has reduced decomposition $w = s_1 \cdots s_k$, let $n(w) = n(s_1) + \cdots + n(s_k)$. Thus $n(w) = \dim E_w$ and so in particular is independent of the choice of reduced decomposition. Now whenever a space has a locally finite cell decomposition, we have a *cell series* $\sum a_i t^i$, where $a_i$ is the number of cells of dimension $i$. We then have:

(2.23) COROLLARY. $G/P_I$ *admits a* $CW$*—decomposition with cell series* $\sum_{w \in W^I} t^{n(w)}$. $\square$

Note also:

(2.24) COROLLARY. *If* $W$ *is finite with maximal length element* $w_0, \mathscr{B}_G$ *is a sphere of dimension* $n(w_0) + l - 1$. $\square$

We conclude this section with two "classical" examples. Let $G$ be a semisimple compact Lie group and consider the Tits system $(G, B, N, S)$, where $B$ is a Borel subgroup, etc. First we claim that this is a topological Tits system satisfying all four of our axioms. Since $W$ is finite, (2.11) is trivially satisfied. In (2.12) we can take $A_s$ to be the "little $SU(2)$" (or $PSU(2)$) $G_s$ ($P_s$ has Iwasawa decomposition $P_s = G_s B$). In any case there is a commutative diagram

$$\begin{array}{ccccc} & & G_s & \rightarrow & P_s \\ & & \downarrow & & \downarrow \\ \mathbf{C}P^1 & = & G_s/G_s \cap T & = & P_s/B \end{array}$$

which proves (2.20), (2.21), and hence (2.12) simultaneously. The Bruhat decomposition of $G_{\mathbf{C}}/P_I$, $P_I$ parabolic, is then the classical Schubert cell decomposition of the flag variety $G_{\mathbf{C}}/P_I$. We have $n(s) = 2$ for all $s$, so $n(w) = 2l(w)$ for all $w \in W^I$. In particular the associated building $\mathscr{B}_{G_{\mathbf{C}}}$ is a sphere of dimension $2l(w_0) + l - 1$ (since $l(w)_0$ is the number of positive roots, this is exactly dim $G - 1$).

The second example (which is a generalization of the first) involves symmetric spaces $G/K$ and the associated semisimple real Lie group $G_{\mathbf{R}}$ as in § 1. Thus $G_{\mathbf{R}}$ is the fixed group of the involution $\sigma$ on $G_{\mathbf{C}}$. Now $\sigma$ need not preserve the Borel subgroup $B$ of $G_{\mathbf{C}}$, but it does preserve the parabolic $Q$ associated to the black nodes of the Satake diagram. We will write $B_{\mathbf{R}}$, $N_{\mathbf{R}}$, $W_{\mathbf{R}}$, $S_{\mathbf{R}}$ for $Q^\sigma$, $N_K t_m$, $W_{G/K}$, $S_{G/K}$, respectively.

(2.25) THEOREM. ($G_{\mathbf{R}}, B_{\mathbf{R}}, N_{\mathbf{R}}, S_{\mathbf{R}}$) *is a topological Tits system satisfying the four axioms.* □

A proof that this is a Tits system can be found in [33]. The parabolic subgroups of $G_{\mathbf{R}}$ are related in an obvious way to those of $G_{\mathbf{C}}$: Given $I \subset S_{\mathbf{R}}$, let $I'$ be the corresponding set in $S$ (see § 1). We denote by $\mathcal{O}_I$ the parabolic in $G_{\mathbf{R}}$ generated by $B_{\mathbf{R}}$ and $I$. Then $\mathcal{O}_I = (P_{I'})^\sigma$. ($B_{\mathbf{R}}$ is usually called a "minimal parabolic", but this terminology conflicts with our use of the term. From the point of view of Tits systems, it is precisely analogous to the Borel subgroup of $G_{\mathbf{C}}$—although in general it is neither solvable nor connected.) The rest of the theorem is also easily deduced from [33]; the details will be omitted, but see § 5. The main point is that for the minimal parabolics $\mathcal{O}_i$, $\mathcal{O}_i/B_{\mathbf{R}}$ is a sphere of dimension $n_i$.

As for the building, one can deduce from (2.24) that it is a sphere whose dimension is dim $G/K - 1$. However it is an interesting fact, that does not seem to appear in the literature, that the building can be canonically identified with the "tangent cut locus" of $G/K$: first recall (cf. [10], [20]) that if $M$ is a compact Riemannian manifold and $p$ is a fixed point of $M$, a point

$x$ is a *cut point* (with respect to $p$) if there is a geodesic from $p$ to $x$ that minimizes arc length up to $x$ but no further. The *cut locus* is the set of cut points. Similarly a vector $X$ in the tangent space $T_p$ is a tangent *cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in $T_p$, and is homeomorphic to the unit sphere in $T_p$. When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let $G/K$ be a simply-connected symmetric space, with $G$ simple. Then the tangent cut locus is precisely the K-orbit in $\mathfrak{m}$ of the outer wall of the Cartan simplex $\Delta_m$. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption $G$ simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathscr{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_m \times \Delta_0$, where $\Delta_0$ is a simplex of dimension (rank $G/K$)-1; we take $\Delta_0$ to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leqslant S_{G/K}$, let $\Delta_I$ temporarily denote the corresponding face of $\Delta_0$; *i.e.* $\{X \in \Delta_0 : \alpha_i(x) = 0 \; \forall \, i \in I\}$. Then the $K$-orbit of $\Delta_0$ in $\mathfrak{m}$, $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \mathring{\Delta}_I$ and $k_1 = k_2 \bmod K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. $\square$

## § 3. LOOP GROUPS

Let $LG$, $LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$, and let $L_{alg}G = L_{alg}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{alg}G$ consists of the loops $f$ in $LG$ admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^{m} A_k z^k$, whereas $L_{alg}G_{\mathbf{C}}$ consists of the loops $f$ in $LG_{\mathbf{C}}$ such that both $f$ and $f^{-1}$ admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{alg}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \to g_{\mathbf{C}}$. The integer $m$ in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let $P$ denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let $\tilde{B}$ denote the Iwahori subgroup: $\{f \in P : f(0) \in B^-\}$. Finally, let $\tilde{N} = L_{alg}N_{\mathbf{C}}$, and recall that $\tilde{W}$ can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leqslant \operatorname{Hom}(S^1, T) \leqslant L_{alg}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.